A GENERAL FATOU LEMMA
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SETUP

Let $\Omega$ be a non-empty internal set, $\mathcal{A}_0$ an internal algebra on $\Omega$, and $\mathcal{A}$ the $\sigma$-algebra generated by $\mathcal{A}_0$.

Let $J$ be a finite or countably infinite set. \(\forall j \in J\), let \((\Omega, \mathcal{A}_0, \mu_{0,j})\) and \((\Omega, \mathcal{A}, \mu_j)\) be internal and Loeb probability spaces.

From these generate $\bar{\mu}$ so that $\forall j$, $\mu_j \ll \bar{\mu}$. We may assume $\mathcal{A}$ is $\bar{\mu}$-complete.

Let $Y$ be a separable Banach lattice, and $X$ is its dual Banach space with the natural dual order (denoted by $\leq$) and lattice norm (i.e., $|x| \leq |z| \Rightarrow \|x\| \leq \|z\|$).

Let $P$ be any probability measure on $(\Omega, \mathcal{A})$. 

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Definition. A sequence \( \{g_n\}_{n=1}^{\infty} \) of functions from \((\Omega, \mathcal{A}, P)\) to \(X\) is said to be weak* \(P\)-tight, if for any \(\varepsilon > 0\), there exists a weak* compact set \(K\) in \(X\) such that for all \(n \in \mathbb{N}\), \(P(g_n^{-1}(K)) > 1 - \varepsilon\).

Definition. For each \(x \in X\), \(y \in Y\), the value of the linear functional \(x\) at \(y\) will be denoted by \(\langle x, y \rangle\). A function \(f\) from \((\Omega, \mathcal{A}, P)\) to \(X\) is said to be Gelfand \(P\)-integrable if for each \(y \in Y\), the real-valued function \(\langle f(\cdot), y \rangle\) is integrable on \((\Omega, \mathcal{A}, P)\).

Proposition. If \(f : (\Omega, \mathcal{A}, P) \rightarrow X\) is Gelfand \(P\)-integrable, then there is a unique \(x \in X\) such that \(\langle x, y \rangle = \int_\Omega \langle f(\omega), y \rangle \, P(d\omega)\) for all \(y \in Y\). (That element \(x\), called the Gelfand integral, will be denoted by \(\int_\Omega f \, dP\).)
Proof. (Well-known): Let $T(y)$ be the element of $L^1(P)$ given by $\omega \mapsto \langle f(\omega), y \rangle$. By Closed Graph Theorem, $\|T\| < \infty$, so

\[
\left| \int_{\Omega} \langle f(\omega), y \rangle \, P(d\omega) \right| \leq \int_{\Omega} |\langle f(\omega), y \rangle| \, P(d\omega) \leq \|T\| \|y\|. \quad \Box
\]

Simplifying Assumption: \exists an increasing (perhaps constant) sequence $y_m \geq 0$ in $Y$ with $\lim_{m \to \infty} \langle x, y_m \rangle = \|x\| \ \forall x \geq 0$ in $X$.

The assumption is valid when $X = \ell^1$ or $X = \mathcal{M}(S)$, the space of finite, signed Borel measures on a second-countable, locally compact Hausdorff space $S$.

The main result, stated here for a sequence of functions $g_n \geq 0$, is generalized with the assumption that each $n \in \mathbb{N}$, $g_n \geq f_n$ where the sequence $\langle f_n \rangle$ has appropriate properties.
Theorem. Let \( \{g_n\}_{n=1}^{\infty} \) be a sequence of nonnegative functions from \( \Omega \) to \( X \).

Suppose \( \forall j \in J \), each function \( g_n \) is Gelfand integrable on \( (\Omega, \mathcal{A}, \mu_j) \), and the Gelfand integrals \( \int_{\Omega} g_n d\mu_j \) have a weak* limit \( a_j \in X \) as \( n \to \infty \).

Then \( \exists g : \Omega \leftrightarrow X \) such that

1. for \( \bar{\mu} \)-a.e. \( \omega \in \Omega \), \( g(\omega) \) is a weak* limit point of \( \{g_n(\omega)\}_{n=1}^{\infty} \),

2. the function \( g \) is Gelfand \( \mu_j \)-integrable with \( \int_{\Omega} g \, d\mu_j \leq a_j \) for each \( j \in J \);

3. the integral \( \int_{\Omega} \langle g, y \rangle \, d\mu_j = \langle a_j, y \rangle \) for any \( y \in Y \) and \( j \in J \) for which \( \{\langle g_n, y \rangle\}_{n=1}^{\infty} \) is uniformly \( \mu_j \)-integrable;

4. In particular, \( \int_{\Omega} g \, d\mu_j = a_j \) for any \( j \in J \) for which the sequence \( \{\|g_n\|\}_{n=1}^{\infty} \) is uniformly \( \mu_j \)-integrable.
Corollary. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of \( \mathcal{A} \)-measurable functions from \( \Omega \) to a complete separable metric space \( Z \).

Assume \( \forall j \in J, \{\mu_j f_n^{-1}\}_{n=1}^{\infty} \) converges weakly to a Borel probability measure \( \nu_j \).

Then, there is an \( \mathcal{A} \)-measurable function \( f \) from \( \Omega \) to \( Z \) such that \( f(\omega) \) is a limit point of \( \{f_n(\omega)\}_{n=1}^{\infty} \) for \( \bar{\mu} \)-a.e. \( \omega \in \Omega \), and \( \mu_j f^{-1} = \nu_j \) for each \( j \in J \).

Corollary. A simplified version of our theorem holds for functions taking values in \( \mathbb{R}^p \), where the norm of each \( x = (x^1, \ldots, x^p) \) in \( \mathbb{R}^p \) is given by \( \sum_{i=1}^{p} |x^i| \).

For a more general theorem, the following consequence of the Simplifying Assumption about \( X \) must be added to the hypotheses.

Claim. \( \forall j \in J, \{g_n\}_{n=1}^{\infty} \) is weak* \( \mu_j \)-tight.
Proof. By an argument of H. Lotz using the Monotone Convergence Theorem,
\[ \forall j \in J, \forall n \in \mathbb{N}, \left| \int_{\Omega} g_n(\omega) d\mu_j \right| \]
\[ = \lim_{m \to \infty} \left< \int_{\Omega} g_n(\omega) d\mu_j, y_m \right> \]
\[ = \lim_{m \to \infty} \int_{\Omega} \left< g_n(\omega), y_m \right> d\mu_j \]
\[ = \int_{\Omega} \lim_{m \to \infty} \left< g_n(\omega), y_m \right> d\mu_j = \int_{\Omega} \| g_n(\omega) \| d\mu_j. \]

The Gelfand integrals \( \int_{\Omega} g_n d\mu_j \) converge in the weak*-topology, so by the Uniform Boundedness Principle \( \exists M_j > 0 \) such that \( \forall n \in \mathbb{N}, \left| \int_{\Omega} g_n d\mu_j \right| \leq M_j. \) Since

\[ \forall n, k \in \mathbb{N}, \int_{\{ \| g_n(\omega) \| \geq k \}} \| g_n \| d\mu_j \leq M_j, \]

\[ \mu_j \left( \{ \omega \in \Omega : \| g_n(\omega) \| \geq k \} \right) \leq M_j/k. \square \]
EXAMPLES
We have an example showing that even for a single measure \( \mu \), there may be no function \( g \) if \( \mu \) is Lebesgue measure on \([0, 1]\).

Here, we let \( X = \ell^1 \). An example of Liapounoff constructs an \( h : [0, 1] \rightarrow \ell^1 \) such that for no \( E \subset [0, 1] \) is it true that for coordinate-wise integration,
\[
\int_E h(t) \, dt = \frac{1}{2} \int_{[0,1]} h(t) \, dt.
\]

We use the Liapounoff Theorem and \( \forall n \) the first \( n \) components of \( h \), to construct a sequence \( g_n \geq 0 \) satisfying the conditions of our theorem, but \( g \) can not exist by the Liapounoff example.

A modification of this first example shows that the corollary, even for \( \mathbb{R}^2 \), can fail when the measures \( \mu_j \) are multiples of Lebesgue measure on \([0, 1]\).
**Lemma 1.** Let $X$ be a standard, separable metric space with metric $\rho$ and the Borel $\sigma$-algebra $\mathcal{B}$. Fix $x_0 \in X$.

Let $P_0$ be an internal probability measure on $(\Omega, \mathcal{A}_0)$ with Loeb space $(\Omega, \mathcal{A}, P)$.

Let $h$ be an internal, measurable map from $(\Omega, \mathcal{A}_0)$ to $(*X,*\mathcal{B})$.

Let $\nu$ be the internal probability measure on $(*X,*\mathcal{B})$ such that $\nu = P_0 h^{-1}$.

Fix a standard tight probability measure $\gamma$ on $(X, \mathcal{B})$ such that $*\gamma \simeq \nu$ in the nonstandard extension of the topology of weak convergence of Borel measures on $X$.

Then the standard part $\circ h(\omega)$ exists for $P$-almost all $\omega \in \Omega$ (where $h(\omega)$ is not near-standard, set $\circ h(\omega) = x_0$). This function $\circ h$ is measurable, and $\gamma = P(\circ h)^{-1}$. 

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Proof. For every standard, bounded, continuous real-valued $f$ on $X$,
\[ \int_{X}^{*} f \, d\nu \simeq \int_{X}^{*} f \, d^*\gamma = \int_{X} f \, d\gamma. \]

Let $K_0 = \emptyset$, and $\forall n \in \mathbb{N}$, let $K_n \supseteq K_{n-1}$ be compact in $X$ with $\gamma(K_n) > 1 - \frac{1}{2n}$. $\forall j \in \mathbb{N}$,
\[ V^j_n := \{ x \in X : \rho(x, K_n) < \frac{1}{j} \} \]
has the property that $\nu(*V^j_n) > 1 - \frac{1}{n}$, whence $\exists H \in *\mathbb{N}_{\infty}$, with $\nu(V^H_n) > 1 - \frac{1}{n}$.

Now the monad $m(K_n) := \cap_{j \in \mathbb{N}} *V^j_n$, and
\[ h^{-1}[m(K_n)] = h^{-1}[\cap_{j \in \mathbb{N}} *V^j_n] = \cap_{j \in \mathbb{N}} h^{-1}[*V^j_n] \]
is measurable and $P(h^{-1}[m(K_n)]) \geq 1 - \frac{1}{n}$.

The standard part $\circ h$ is defined on $h^{-1}[m(K_n)]$, is measurable there, and takes values in $K_n$. 9
Therefore, \( \circ h \) defines a measurable mapping from \( \bigcup_n h^{-1}[m(K_n)] \) to \( \bigcup_n K_n \), and

\[
P \left( \bigcup_n h^{-1}[m(K_n)] \right) = 1.
\]

Set \( \circ h = x_0 \) on \( \Omega \setminus \bigcup_n h^{-1}[m(K_n)] \).

With this extension, \( \circ h \) is a measurable mapping defined on \((\Omega, \mathcal{A}, P)\).

Finally, given a bounded, continuous, real-valued function \( f \) on \( X \),

\[
\int_X f \, dP(\circ h)^{-1} = \int_\Omega f \circ \circ h \, dP
\]

\[
= \int_\Omega \text{st} \left( * f \circ h \right) \, dP
\]

\[
\simeq \int_\Omega * f \circ h \, dP_0 = \int_{*_X} * f \, d
\]

\[
\simeq \int_{*_X} * f \, d* \gamma = \int_X f \, d\gamma.
\]

It follows that \( \gamma = P(\circ h)^{-1} \) on \( X \). \( \square \)

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Lemma 2. Let \((X, \rho)\) be a separable metric space with the Borel \(\sigma\)-algebra \(\mathcal{B}\).

Let \(P_0\) be an internal probability measure on \((\Omega, \mathcal{A}_0)\) with Loeb space \((\Omega, \mathcal{A}, P)\).

Fix an internal sequence \(\{h_n : n \in \mathbb{N}\}\) of measurable maps from \((\Omega, \mathcal{A}_0)\) to \((X, \mathcal{B})\).

Fix a nonempty compact \(K \subseteq X\).

Then \(\exists H \in \mathbb{N}_\infty\) and a \(P\)-null set \(S \subset \Omega\) such that

if \(n \leq H\) in \(\mathbb{N}_\infty\), while \(\omega \notin S\),
and \(h_n(\omega)\) has standard part in \(K\),

then for any standard \(\varepsilon > 0\), there are infinitely many limited \(k \in \mathbb{N}\) for which
\[\ast \rho(h_k(\omega), h_n(\omega)) < \varepsilon.\]
Proof. Given $l \in \mathbb{N}$ cover $K$ with $n_l$ open balls of radius $1/l$. Let $B(l, j)$ denote the nonstandard extension of the $j^{th}$ ball.

For each $i \in \mathbb{N}$, set
\[ A_i(l, j) := \{ \omega \in \Omega : h_i(\omega) \notin B(l, j) \}. \]

For all $k \in \mathbb{N}$, choose $m_k(l, j) \in \mathbb{N}_\infty$ so that
\[ \mathbb{P}\left( \bigcap_{i=k}^{\infty} A_i(l, j) \right) = \mathbb{P}\left( \bigcap_{i=k}^{\infty} A_i(l, j) \right). \]

Set
\[ S_k(l, j) := \left( \bigcap_{i=k}^{\infty} A_i(l, j) \right) \setminus \bigcap_{i=k}^{m_k(l, j)} A_i(l, j). \]

Fix $H \in \mathbb{N}_\infty$ with $H \leq m_k(l, j)$

For all $l \in \mathbb{N}$, $j \leq n_l$, and $k \in \mathbb{N}$.

Let $S$ be the $P$-null set formed by the union of the set $S_k(l, j)$ for all $l \in \mathbb{N}$, $j \leq n_l$, and $k \in \mathbb{N}$. 12
Fix $n \in \, ^*\mathbb{N}_\infty$ with $n \leq H$, and suppose $\text{st} (h_n(\omega)) \in K$ but $\exists l \in \mathbb{N}$ for which there are at most finitely many limited $k \in \mathbb{N}$ for which $^*\rho(h_k(\omega), h_n(\omega)) < 2/l$.

Then for some $j \leq n_l$, $h_n(\omega) \in B(l, j)$, and by assumption there is a limited $k \in \mathbb{N}$ such that for all limited $i \geq k$, $h_i(\omega) \notin B(l, j)$. It follows that $\omega \in S_k(l, j) \subseteq S$. □

**Idea of Parts of Theorem’s Proof.**
Replace sequence $\{g_n\}$ with a subsequence so $\forall j \in J$, $\mu_j g_n^{-1}$ converges weakly to $\gamma_j$.

Lift and extend $\{g_n\}$ to $\{h_n\}$ and work with measures $\mu_{0j} h_n^{-1}$. Use Lemma 1 to show $\exists H \in \, ^*\mathbb{N}_\infty$ so $g(\omega) := (\circ h_H)(\omega)$ exists for $\bar{\mu}$-a.e. $\omega \in \Omega$ and $\gamma_j = \mu_j (\circ h_H)^{-1}$ $\forall j \in J$.

Use Lemma 2 to show that for $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^\infty$. 13