Non-Standard Methods and Reverse Mathematics

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From Hilbert’s Program to Reverse Math.

Hilbert’s Reductionism Program:
Reduce the whole math to finite math.

Reverse Math Program (Friedman-Simpson):
Reduce a stronger system ($\text{WK}L_0$) to a weaker system ($\text{PRA}$).

How much math can be developed within the stronger system?
Practice of Reverse Mathematics

0. Fix a base theory $T \ (\equiv RCA_0)$.

1. Pick a theorem $\tau$.

2. Find the weakest axiom $\alpha$ s.t.

   $T + \alpha \vdash \tau$.

3. Very often, we can show

   $T \vdash \alpha \leftrightarrow \tau$. 
Reverse Mathematics

- Reverse math classifies mathematical theorems, according to which set existence axioms are needed to prove them.

Framework: Second order arithmetic \( \mathbb{Z}_2 \)

- Basic axioms for \((+, \cdot, 0, 1, <)\)
- Comprehension (\(CA\)): \(\exists X \forall x(x \in X \iff \varphi(x))\)
- Induction: \(\varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x \varphi(x)\)
Classes of Formulas

- Bounded formulas \((\Sigma^0_0)\), only with \(\forall x < t, \exists x < t\)

- Arithmetical formulas \((\Sigma^1_0)\), with no set quantifiers

\[
\Sigma^0_n : \exists \overrightarrow{x_1} \forall \overrightarrow{x_2} \cdots Q \overrightarrow{x_n} \varphi \text{ with } \varphi \text{ bounded.}
\]

\[
\Pi^0_n : \forall \overrightarrow{x_1} \exists \overrightarrow{x_2} \cdots Q \overrightarrow{x_n} \varphi \text{ with } \varphi \text{ bounded.}
\]

- Analytical formulas:

\[
\Sigma^1_n : \exists \overrightarrow{X_1} \forall \overrightarrow{X_2} \cdots Q \overrightarrow{X_n} \varphi \text{ with } \varphi \text{ arithmetic.}
\]

\[
\Pi^1_n : \forall \overrightarrow{X_1} \exists \overrightarrow{X_2} \cdots Q \overrightarrow{X_n} \varphi \text{ with } \varphi \text{ arithmetic.}
\]
Big five subsystems

\[ RCA_0 = \Delta^0_1-CA + \Sigma^0_1\text{-ind} \]

\[ WKL_0 = RCA_0 + \text{ weak König's lemma for infinite binary trees} \]

\[ ACA_0 = RCA_0 + \Sigma^0_1-CA \]

\[ ATR_0 = RCA_0 + \text{ iteration of } \Sigma^0_1-CA \text{ along any well ordering} \]

\[ \Pi^1_1-CA_0 = RCA_0 + \Pi^1_1-CA \]
Some results of R. M.

Over $\text{RCA}_0$

$WKL_0 \iff \text{the maximum principle}$
  $\iff \text{the Cauchy-Peano theorem}$
  $\iff \text{Brouwer’s fixed point theorem}$

$\text{ACA}_0 \iff \text{the Bolzano-Weierstrass theorem}$
  $\iff \text{the Ascoli lemma}$

$\text{ATR}_0 \iff \text{the Luzin separation theorem}$
  $\iff \Sigma^0_1$-determinacy

$\Pi^1_1-\text{CA}_0 \iff \text{the Cantor-Bendixson theorem}$
  $\iff \Sigma^0_1 \land \Pi^0_1$-determinacy
Remarks.

- While doing reverse mathematics, one often needs to invent a new proof or modify an old proof for the popular theorem, so that it suits for a weaker subsystem.
- For instance, to prove the Peano existence theorem within $WKL_0$, Simpson (1984) invented a new proof which does not depend on the Ascoli lemma, which is known to be stronger than $WKL_0$.
- Non-standard methods are useful in many cases.
Nonstandard praxis is remarkably constructive.

In the standard approach one uses in the final step the Ascoli lemma. This part of the argument is lacking in the nonstandard proof.

It is possible to recast the nonstandard proof to give a proof of the Peano existence theorem where the only nonrecursive element is the weak Konig’s Lemma.
Non-standard methods

✓ Conservation:

\[ T + \alpha \vdash \varphi \Rightarrow T \vdash \varphi \]

✓ Inner models:

\[ T \vdash (\varphi \leftrightarrow M \models \varphi) \]

✓ Outer models:

\[ M \models \varphi \iff ^*M \models ^*\varphi \]
Conservation results

✓ Shoenfield:

\[ ZF + V = L \vdash \sigma \Rightarrow ZF \vdash \sigma \text{ for } \sigma \in \Sigma_2^1 \cup \Pi_2^1 \]

✓ Barwise-Schlipf:

\[ \Sigma_1^1 - AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma \text{ for } \sigma \in \Pi_2^1 \]

✓ Harrington:

\[ WKLO \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \text{ for } \sigma \in \Pi_1^1 \]


\[ WKLO \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \]

\[ \text{for } \sigma \equiv \forall X \exists ! Y \varphi(X,Y) \text{ with } \varphi \text{ arith.} \]
Application of Simpson-T.-Yamazaki's result

The fundamental theorem of algebra (FTA):

*Any complex polynomial of a positive degree has a unique factorization into linear terms.*

\[ WKL_0 \models \text{FTA with standard polynomials} \]

\[ \therefore RCA_0 \models \text{FTA with standard polynomials} \]

or \[ RCA \models \text{FTA} \]
Inner model methods

- Count. corded $\beta_n$-models and reflection.
- Resplendency and recursive saturation.
- Defining the satisfaction relation on $\mathbb{R}$.
Defining the real number system $\mathbb{R}$

The following definitions are made in $RCA_0$.

✓ Using the pairing function, we define $\mathbb{N}$ and $\mathbb{Q}$.
✓ The basic operations on $\mathbb{N}$ and $\mathbb{Q}$ are also naturally defined.
✓ A real number is an infinite sequence $\{q_n\}$ of rationals such that $|q_n - q_m| \leq 2^{-n}$ for all $m > n$.
✓ The operations on $\mathbb{R}$ are also defined so that the resulting structure is a real closed order field.
Satisfaction on $\mathbb{R}$

✓ Simpson-T.-Yamazaki

$Sat_{\mathbb{R}}([\varphi(\vec{x})], \vec{\xi})$ can be defined as a $\Delta^0_2$ formula.

In $RCA_0$, $Sat_{\mathbb{R}}$ satisfies the Tarski clauses for the standard formulas.


In $RCA_0$, $Sat_{\mathbb{R}}$ satisfies the Tarski clauses for all the formulas. In particular,

$Sat_{\mathbb{R}}([\exists \vec{x} \varphi(\vec{x}, \vec{y})], \vec{\beta}) \leftrightarrow \exists \vec{\alpha} Sat_{\mathbb{R}}([\varphi(\vec{x}, \vec{y})], \vec{\alpha}, \vec{\beta})$

♦ The following fact (called strong FTA) is essential:

$RCA_0 \vdash \forall p(x) \in \mathbb{Q}[x] \exists \vec{\alpha} \in \mathbb{C}^{\mathbb{N}} p(x) = \prod_i (x-\alpha_i)$
Applications of Sakamoto-T’s result

\[ RCA_0 \vdash \text{Hilbert’s Nullstellensatz:} \]
\[ p_1, \cdots, p_m \in \mathbb{C}[\overrightarrow{x}] \text{ have no common zeros} \]
\[ \Rightarrow \exists q_1 \cdots \exists q_m \in \mathbb{C}[\overrightarrow{x}] \ p_1q_1 + \cdots + p_mq_m = 0 \]

\[ RCA_0 \vdash \text{strong FTA} \]
Outer Model Method

Theorem (H. Friedman, Kirby-Paris)

Suppose \( M \models PRA \), countable.

Suppose \( b \ll_M c \) (i.e., \( f(b) <_M c \) for all prim. rec. \( f \)).

Then \( \exists I \subseteq_e M \) s.t. \( b \in I, c \notin I \) and \( I \models \text{I} \Sigma_1 \).

Moreover, if \( C(M) = \{ X \subseteq M : \exists a \in M \text{ codes } X \} \),

\( (I, C(M) \upharpoonright I) \models WKL_0 \).

Theorem (T.) A converse to the above holds.

Suppose \( (M, S) \models WKL_0 \), countable, \( M \neq \omega \).

Then \( \exists^* M \supseteq_e M \) s.t. \( ^* M \models \text{I} \Sigma_1 \) and \( S = C(M^*) \upharpoonright M \).
Self-Embedding Theorems

Thm. (self-embedding for $\text{WKL}_0$, T. 1997)

Suppose $(M, S) \models \text{WKL}_0$, countable, $M \not= \omega$.

Then $\exists I \subsetneq M \text{ s.t. } (M, S) \sim (I, S \cap I)$.

- History of self embedding results.
  
  \textit{H.Friedman (1970's) for PA.}
  
  \textit{Ressayre, Dimitracopoulos and Paris (1980's) for } I\Sigma_1.

(Proof) By a back-and-forth argument.

Cor. Suppose $(M, S) \models \text{WKL}_0$, countable, $M \not= \omega$.

Then $\exists^* M \subsetneq M, \exists^* S \text{ s.t. } (*M, *S) \models \text{WKL}_0$

and $S = *S \upharpoonright M$. 
Application (the maximum principle)

WKL₀ ⊢ Any cont. function \( f : [0, 1] \rightarrow [0, 1] \) has a max.

(Proof) \[ V = (M, S) \quad \Rightarrow \quad *V = (*M, *S) \]

<table>
<thead>
<tr>
<th>( f : [0, 1] \cap \mathbb{Q} \rightarrow [0, 1] )</th>
<th>( {q_i}_{i \in M} \rightarrow 2^M )</th>
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<tbody>
<tr>
<td>( {q_i}_{i &lt; a} \rightarrow 2^b )</td>
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\( a, b \in *M - M, f = *f \cap M \)

\( *m \cap M \) is sup \( f \) \( \iff *m = \max\{*f(q_i)\}_{i < a} \)
Application (New)

\[ WKL_0 \vdash \text{Strong FTA.} \]

(Proof) \[ V = (M, S) \]

\[ f : \mathbb{Q}[x] \to (\mathbb{C} \cap \mathbb{Q}^2)^{<\mathbb{N}} \]

\( \{p_i\}_{i \in M} \) with infinite repetition

s.t. \( f(p_i) \) is a list of rational approximations of the roots of \( p_i \) with error \(< 2^{-i} \).

\[ \star f(p_{j_i}) \Gamma M \text{ is the list of roots of } p_i. \]

\[ \{p_i\}_{i \in M} = \{p_{j_i}\}_{j_i \not\in M, i \in M} \]

\[ \star f : \{p_i\}_{i < a} \to (\star \mathbb{C} \cap \star \mathbb{Q}^2)^{<b} \]

\( (a, b \in \star M - M, f = \star f \cap M) \)
Other applications

$\textit{WKL}_0 \vdash \text{The Cauchy–Peano theorem} \ (\text{Tanaka, 1997})$

$\textit{WKL}_0 \vdash \text{The existence of Haar measure for a compact group} \ (\text{Tanaka-Yamazaki, 2000})$

$\textit{WKL}_0 \vdash \text{The Jordan curve theorem} \ (\text{Sakamoto-Yokoyama, to appear})$
Application (Sakamoto, Yokoyama)

\[ WKL_0 \vdash \text{The Jordan Curve Theorem} \]

(Proof) \hspace{1cm} V = (M, S) \hspace{1cm} *V = (*M, *S)

\[ U_1 \]

\[ U_0 \]

\[ *U_1 \]

\[ *U_0 \]
Outer model method for $ACA_0$

Suppose $(M, S) \models ACA_0$, countable, $M \neq \omega$.

Then $\exists^* M \supsetneq_e M \exists^* S$

s.t. $(^* M, ^* S) \models ACA_0$, $S = ^* S \upharpoonright M$

and $\exists^* : S \to ^* S \ \forall \varphi(x, X) \in \Sigma^1_1 \cup \Pi^1_1$

$(M, S) \models \varphi(m, A) \leftrightarrow (^* M, ^* S) \models \varphi(m, ^* A)$

This easily follows from

Theorem (Gaifman): Every model $M$ of $PA$ has a conservative extension $K$, i.e., (the sets definable in $K$) $\upharpoonright M = \text{the sets definable in } M$. 
Applications

\[ \text{ACA}_0 \vdash \text{Any Cauchy sequence converges.} \]

(Proof) \quad V = (M, S) \quad \Rightarrow \quad *V = (*M, *S)

\[
\{a_i\}_{i \in M} \text{ a Cauchy seq.} \quad \Rightarrow \quad *
\{(a_i)_{i \in M}\} = \{(a_i)_{i \in *M}\}.
\]

\[
\forall n \exists m \forall k > m |a_k - b| < 2^{-n} \quad \iff \quad |(a_k) - (a_j)| < 2^{-n}.
\]

\[
\forall n \exists m \forall k > m \quad Pick \ j \in *M - M.
\]

\[
(n) \exists m \in M \forall k > m \quad (Yokoyama, \ \text{to appear})
\]
THANK YOU