ON THE WAVE EQUATION WITH A LARGE ROUGH POTENTIAL

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Abstract. We prove an optimal dispersive $L^\infty$ decay estimate for a three dimensional wave equation perturbed with a large non smooth potential belonging to a particular Kato class. The proof is based on a spectral representation of the solution and suitable resolvent estimates for the perturbed operator.

1. Introduction

A fundamental property of solutions to the wave equation for $n \geq 2$

\begin{equation}
\Box^{1+n} u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = f(x)
\end{equation}

is expressed by the dispersive estimate

\begin{equation}
\|u(t, \cdot)\|_{L^\infty} \leq C t^{-\frac{n+1}{2}} \|f\|_{\dot{B}^{\frac{n-1}{2}}_{1,1}(\mathbb{R}^n)}.
\end{equation}

Here $\dot{B}^{s}_{p,q}(\mathbb{R}^n)$ is the homogeneous Besov space whose norm is defined as follows:

\begin{equation}
\|f\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} 2^{jsq} \|\phi_j(\sqrt{-\Delta}) f\|_{L^p},
\end{equation}

where $\phi_j(r) = \phi_j(|x|)$ is a homogeneous Paley-Littlewood decomposition, $\phi_j(r) = \phi_0(2^{-j} r)$, with $\phi_0(r) = \psi(r) - \psi(r/2)$, $\psi(r)$ a nonnegative function in $C_0^\infty$ and $\psi(r) = 1$ for $r < 1$, $\psi(r) = 0$ for $r > 2$.

From (1.2) and the energy identity, via interpolation and the $T^*T$ method, the full set of decay estimates for the wave equation can be derived (see [9] and [16]). The importance of decay estimates for the applications to nonlinear problems is well known and we shall not discuss it here.

The extension of (1.2) to wave equations perturbed with a potential is a very interesting problem, although a quite difficult one. Indeed, potential perturbations like

\begin{equation}
\Box^{1+n} u + V(x) u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = f(x),
\end{equation}

are frequently encountered when studying the stability of stationary solutions for several important systems of partial differential equations (wave-Schrödinger, Maxwell-Schrödinger, Maxwell-Dirac and many others). The potentials that arise are usually non-smooth, and this is one of the main motivations for considering rough functions $V(x)$ in (1.4). This problem has been investigated in many papers (see e.g. [3], [4], [5], [6], [7], [18], and see also, for the closely connected problem concerning the Schrödinger equation, [14], [19], [26], [27]).

The proof of (1.2) is based either on the explicit expression of the fundamental solution, or on the method of stationary phase (see e.g., [21]). Neither method is available in presence of a general potential, although ideas from the second one can

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be adapted to the more general situation. Beals and Strauss [4] proved \( L^p = L^{p'} \)
decay estimates of Strichartz type, later improved by Beals [3] who could prove an
almost optimal dispersive estimate like (1.2), for smooth positive potentials
decaying fast enough at infinity (and \( n \geq 3 \)). Their method is based on a repeated
use of Duhamel’s formula and an explicit kernel representation of the operators that
arise. The same methods permit to handle the case of small potential changing sign
(but still smooth and rapidly decaying).

In the special, and most important case of dimension \( n = 3 \) there were some
essential improvements. Cuccagna [6], using distorted Fourier transform methods,
was able to handle potentials decaying with two derivatives like \( |x|^{-3-\varepsilon} \) at infinity.
The best result in this direction was obtained by Georgiev and Visciglia [7] who
were able to prove the dispersive estimate (1.2) for \( n = 3 \) and potentials of H"older
class \( V(x) \in C^\alpha(\mathbb{R}^3 \setminus 0) , \alpha \in ]0,1[ \), satisfying for some \( \varepsilon > 0 \)
\[
|V(x)| \leq \frac{C}{|x|^{s+\varepsilon} + |x|^{2-\varepsilon}}.
\]
Notice that this implies \( V \in L^{3/2-\delta} \cap L^{3/2+\delta} \) for \( \delta \) small \((0 < \delta < 3\varepsilon/4)\). This
decay assumption on the potential \( V(x) \) is close to optimal, at least in view of the
dispersive estimate. Indeed, in [18] and [5], the “limit” case \( V = a|x|^2 \) was
considered; the authors were able to prove the full set of mixed space-time Strichartz
estimates, but they also gave strong evidence that the dispersive estimate may be
false (although some substitute weighted estimate can be proved). We remark that
this potential belongs to the weak \( L^{3/2}_w \simeq L^{3/2,\infty} \) Lorentz space.

Thus it is natural to ask what are the weakest assumptions on the potential that
imply the dispersive estimate. In the present work we further improve the result of
[7] by proving that it is sufficient to assume that \( V \) belongs to a suitable Kato class
of potentials, while no smoothness is required. Before stating our result we recall
the relevant definitions:

**Definition 1.1.** The measurable function \( V(x) \) on \( \mathbb{R}^n \), \( n \geq 3 \), is said to belong to the
Kato class if
\[
\lim \sup_{r \to 0} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0.
\]
Moreover, the Kato norm of \( V(x) \) is defined as
\[
||V||_K = \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy.
\]
For \( n = 2 \) the kernel \( |x-y|^{2-n} \) is replaced by \( \log(|x-y|^{-1}) \).

Notice that a compactly supported function of Kato class has a finite Kato norm
(see Lemma 4.4).

**Remark 1.1.** The relevance of the Kato class in the study of Schrödinger
operators is well known; full light on its importance was shed by Barry Simon [22] and
Aizenmann and Simon [2]. The stronger norm (1.6) was introduced by Rodnianski
and Schlag [20] who proved the dispersive estimate for the three dimensional
Schrödinger equation with a potential having both the Kato and the Rolnik norms
small.

In connection with dispersive estimates, it is important to recall the definition of resonances. A *resonance* is a point of the spectrum such that there exists \( f \in L^2_{loc} \),
with \((1 + |x|^2)^{-s/2} f \in L^2 \) for some \( s > 1/2 \), satisfying \((\lambda - \Delta + V)f = 0\) in
distribution sense. If a resonance is present on the positive real axis, the dispersive
estimate cannot hold in general; this (and in part our method of proof) is the
motivation of assumption (1.9) in our result, which more explicitly means: if \( f \in \)
$L^2_{loc}$ with $(1 + |x|^{2})^{-s/2} f \in L^2$ for some $s > 1/2$ satisfies $(\lambda - \Delta + V)f = 0$ for some $\lambda$ in a suitable neighbourhood of the spectrum, then $f \equiv 0$.

Our result is then the following:

**Theorem 1.1.** Let $V(x) = V_+ - V_-$, with $V_\pm \geq 0$, be a potential in the Kato class satisfying

\begin{equation}
\|V_+\|_K < \infty, \quad \|V_-\|_K < 2\pi.
\end{equation}

Moreover, denoting with $\chi_R(x)$ the characteristic function of the ball $B(0, R)$ in $\mathbb{R}^3$, and writing $V_R = V(x)\chi_R(x)$, we assume that for some $R > 0$

\begin{equation}
\|V - V_R\|_K < \frac{4\pi}{\|V_R\|_K}.
\end{equation}

Finally, we assume that

\begin{equation}
-\Delta + V \text{ has no resonances in some complex neighbourhood of } \mathbb{R}.
\end{equation}

Then any solution $u(t, x)$ to the Cauchy problem (1.4) satisfies the dispersive estimate

\begin{equation}
\|u(t, \cdot)\|_{L^\infty} \leq C t^{-1}\|f\|_{B^1_{2,1}(\mathbb{R}^3)}.
\end{equation}

**Remark 1.2.** By the extended Young inequality, the Kato norm is bounded by the $L^{3/2,1}$ Lorentz norm:

$$
\|V\|_K \leq C_0 \|V\|_{L^{3/2,1}(\mathbb{R}^3)}.
$$

A simpler but weaker estimate follows directly from H"older inequality: for any small $\delta > 0$ we have

$$
\|V\|_K \leq C(\delta) (\|V\|_{L^{3/2-\delta}(\mathbb{R}^3)} + \|V\|_{L^{3/2+\delta}(\mathbb{R}^3)}).
$$

Thus a consequence of Theorem 1.1 is that the dispersive estimate holds for all potentials $V$ in $L^{3/2,1}(\mathbb{R}^3)$ such that

$$
\|V_+\|_{L^{3/2,1}} < 2\pi C_0^{-1};
$$

or also for any potential $V$ such that, for some $\delta > 0$, $V$ is in $L^{3/2-\delta}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3)$ and

$$
\|V_+\|_{L^{3/2-\delta}(\mathbb{R}^3)} + \|V_-\|_{L^{3/2+\delta}(\mathbb{R}^3)} \leq 2\pi C(\delta)^{-1}
$$

Hence we see that our result extends all the above mentioned papers; in the scale of Lorentz spaces we can now say that the dispersive estimate holds in $L^{3/2,1}$ but not in $L^{3/2,\infty}$. It is not clear what can be said for potentials of Lorentz class $L^{3/2,q}$ with $1 < q < \infty$; it would be interesting to close this gap, but probably different techniques are required.

**Remark 1.3.** The most technical part of the paper is Section 5 where we show the equivalence of homogeneous Besov norms

$$
\dot{B}^s_{1,q}(\mathbb{R}^n) \equiv \dot{B}^s_{1,q}(V), \quad 0 < s < 2, \quad 1 \leq q \leq \infty
$$

for all potentials $V = V_+ - V_-$ with $V_\pm \geq 0$ and

\begin{equation}
\|V_+\|_K < \infty, \quad \|V_-\|_K < \frac{1}{2} C_0 \equiv \pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right)
\end{equation}

(Theorem 5.6). By the way, this is the only step where (1.7) is needed; if one is satisfied with the dispersive estimate in terms of the $\dot{B}^1_{1,1}(V)$ norm of $f$, then the more natural assumption

\begin{equation}
\|V_+\|_K < \infty, \quad \|V_-\|_K < 4\pi
\end{equation}

is sufficient (also in order to have a self-adjoint positive operator). In order to prove this result, which has maybe an independent interest, we needed to extend
some lemmas of [12]-[13] which in turn forced us to improve some estimates for the Schrödinger semigroup due to Simon [22]. Indeed, in Proposition 5.1 we prove that the semigroup \( e^{t(\Delta - V)} \) has an integral kernel \( k(t, x, y) \) satisfying
\[
|k(t, x, y)| \leq \frac{(2\pi t)^{-n/2}}{1 - 2\|V_\|_K/c_n} e^{-|x-y|^2/4t}
\]
and satisfies the estimate
\[
\|e^{-tH}\|_{L(L^p, L^q)} \leq \frac{(2\pi t)^{-\gamma}}{(1 - \|V_\|_K/c_n)^2}, \quad \gamma = \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Thus an immediate, and unexpected, byproduct of our method of proof is the following parabolic dispersive estimate (see Proposition 5.1):

**Theorem 1.2.** Let \( n \geq 3 \), assume the potential \( V(x) \) is of Kato class, has a finite Kato norm and its negative part \( V^- \) satisfies
\[
\|V^-\|_K < 2\pi^{n/2}/\Gamma \left( \frac{n}{2} - 1 \right)
\]
Then the solution \( u(t, x) \) to the perturbed heat equation
\[
(1.16) \quad u_t - \Delta u + V(x)u = 0, \quad u(0, x) = f(x)
\]
satisfies the dispersive estimate
\[
(1.17) \quad \|u(t, \cdot)\|_{L^q} \leq C t^{-n/2p} \|f\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad q \in [1, \infty].
\]

**Remark 1.4.** A key remark from [7] is that in dimension \( n = 3 \) the spectral representation of the solution and an integration by parts are sufficient to prove the dispersive estimate, provided suitable \( L^1 - L^\infty \) estimates for the spectral measure are available.

Here we follow the same line of proof, with an improvement: indeed, it is not necessary to compute an explicit representation of the spectral measure as an operator (i.e., to prove the *limiting absorption principle* for \(-\Delta + V\)), but it is sufficient to apply the spectral theorem directly and prove uniform estimates for the resolvent operator outside the spectrum. This idea was used in a previous work [17] where the case of potentials with small Kato norm was studied.

The paper is organized as follows. In Section 2 we recall some basic properties of the free resolvent and prove some uniform estimates for it. Section 3 is devoted to a study of the operator \(-\Delta + V\). In Section 4 we prove the crucial estimates for the spectral measure. Section 5 contains the above mentioned study of the Schrödinger semigroup, which is then the applied to estimate functions of the operator \( f(-\Delta + V) \), and to prove the equivalence of free and perturbed Besov spaces. Finally, in Section 6 the estimates of Section 4 are applied to the spectral representation of the solution, thus obtaining a dispersive estimate with perturbed Besov norm; in combination with the equivalence result of Section 5, this concludes the proof.

### 2. Properties of the free resolvent

We start from the explicit representation of the free resolvent \( R_0(z) = (-\Delta - z)^{-1} \) (see e.g. [19]), which is usually expressed in terms of the square of the parameter:
\[
(2.1) \quad R_0(\xi^2)g(x) = (-\Delta - \xi^2)^{-1}g = \begin{cases} 
\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|\xi| |x-y|}}{|x-y|} g(y) dy & \text{for } \text{Im } \xi > 0 \\
\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-i|\xi| |x-y|}}{|x-y|} g(y) dy & \text{for } \text{Im } \xi < 0.
\end{cases}
\]
By elementary computations we obtain that for any \( \lambda \in \mathbb{R} \) and \( \varepsilon > 0 \)

\[
R_0(\lambda \pm i\varepsilon)g(x) = \frac{1}{4\pi} \int e^{\mp \sqrt{\lambda} |x-y|} e^{-\varepsilon|x-y|^2/2\sqrt{\lambda}} g(y) dy
\]

where

\[
\lambda_\varepsilon = \frac{\lambda + (\lambda^2 + \varepsilon^2)^{1/2}}{2} > 0.
\]

These formulas define bounded operators on \( L^2 \), provided \( \varepsilon > 0 \). When approaching the positive real axis, i.e., as \( \varepsilon \downarrow 0 \), this property fails; nevertheless if we define the limit operators for \( \lambda \geq 0 \)

\[
R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi} \int e^{\mp \sqrt{\lambda} |x-y|} g(y) dy
\]

then it is possible to prove that \( R_0(\lambda \pm i0) \) are bounded from the weighted space \( L^2(|x|^s \, dx) \) to \( L^2(|x|^{-s} \, dx) \) for any \( s > 1/2 \), and actually \( R_0(\lambda \pm i\varepsilon) \rightarrow R_0(\lambda \pm i0) \) in the operator norm. This is sometimes referred to as the **limiting absorption principle** (see [1], [11]).

For **negative** \( \lambda \) we have quite strong estimates (indeed, the negative real axis belongs to the resolvent set of \( -\Delta \)): since

\[
0 < \lambda_\varepsilon < \frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}} \geq \sqrt{|\lambda|} \quad \text{for all } \lambda < 0
\]

we have from (2.2), for all \( \lambda < 0, \varepsilon \geq 0 \)

\[
|R_0(\lambda \pm i\varepsilon)g(x)| \leq \frac{1}{4\pi} \int e^{-\sqrt{|\lambda|} |x-y|} |g(y)| dy
\]

and actually for \( \lambda < 0, \varepsilon = 0 \)

\[
R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi} \int e^{-\sqrt{|\lambda|} |x-y|} g(y) dy.
\]

We collect here some inequalities which follow immediately from the above representations and will be useful in the following computations. Since

\[
[R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)]g = \frac{i}{2\pi} \int \sin(\sqrt{\lambda_\varepsilon} |x-y|) e^{-\varepsilon|x-y|^2/2\sqrt{\lambda}} g(y) dy
\]

we can write for all \( \lambda \in \mathbb{R} \) and \( \varepsilon \geq 0 \)

\[
\| [R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)] g \|_{L^\infty} \leq \frac{\sqrt{\lambda_\varepsilon}}{2\pi} \| g \|_{L^1}.
\]

Recalling Definition 1.1 we see by a straightforward computation that

\[
\| R_0(\lambda \pm i\varepsilon) V g \|_{L^\infty} \leq \frac{1}{4\pi} \| V \|_K \| g \|_{L^\infty} \quad \forall \lambda \in \mathbb{R}, \varepsilon \geq 0
\]

for any measurable function \( V(x) \), and that

\[
\| VR_0(\lambda \pm i\varepsilon) g \|_{L^1} \leq \frac{1}{4\pi} \| V \|_K \| g \|_{L^1} \quad \forall \lambda \in \mathbb{R}, \varepsilon \geq 0.
\]

Of course for **negative** \( \lambda \) we have better estimates:

**Lemma 2.1.** Assume \( \| V \|_K < \infty \). Then for all \( \delta > 0 \) there exists \( C_\delta > 0 \) such that

\[
\| R_0(\lambda \pm i\varepsilon) V g \|_{L^\infty} \leq \left( \delta + C_\delta \frac{\| V \|_K}{\sqrt{|\lambda|}} \right) \| g \|_{L^\infty} \quad \forall \lambda < 0, \varepsilon \geq 0
\]
and
\[ \| VR_0(\lambda \pm i\varepsilon)g \|_{L^1} \leq \left( \delta + C_3 \frac{\| V \|_K}{\sqrt{|\lambda|}} \right) \| g \|_{L^1}, \quad \forall \lambda < 0, \varepsilon \geq 0. \]

**Proof.** By (2.5) we have
\[ |R_0(\lambda \pm i\varepsilon)Vg(x)| \leq \frac{1}{4\pi} \int \frac{|V(y)|}{|x-y|} |g(y)| e^{-\sqrt{|\lambda|}|x-y|} dy. \]

Now for any \( r > 0 \) we can split the integral in two zones \( |x-y| < r \) and \( |x-y| \geq r \); for the first piece we have
\[ \frac{1}{4\pi} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|} |g(y)| e^{-\sqrt{|\lambda|}|x-y|} dy \leq \frac{1}{4\pi} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|} |g(y)| \| g \|_{L^\infty} \]
and this can be made smaller than \( \delta \| g \|_{L^\infty} \) by the definition of Kato class (1.5), provided we choose \( r < r(\delta) \) (recall also that a function with finite Kato norm belongs to the Kato class: see Lemma 4.4). With this choice we can estimate the second piece as follows
\[ \frac{1}{4\pi} \int_{|x-y| \geq r} \frac{|V(y)|}{|x-y|} |g(y)| e^{-\sqrt{|\lambda|}|x-y|} dy \leq \frac{\| g \|_{L^\infty}}{4\pi r \sqrt{|\lambda|}} \int \frac{|V(y)|}{|x-y|} dy \]
where we have used the inequality \( e^{-a} \leq 1/a \), and this proves (2.10). Estimate (2.11) follows by an analogous proof, or in a simpler way duality. \( \square \)

We shall also need estimates for the square of the resolvent \( R_0(\lambda \pm i\varepsilon)^2 \). Since by the resolvent identity
\[ \frac{d}{dz} R_0(z) = R_0(z), \]
we have the explicit representations
\[ R_0(\lambda \pm i\varepsilon)^2 g = \frac{1}{8\pi} \left( \pm \sqrt{\lambda} \pm i \frac{\varepsilon}{2\sqrt{\lambda}} \right)^{-1} \int e^{\pm i \sqrt{|\lambda|} |y-x|} g(y) dy \]
and
\[ R_0(\lambda \pm i0)^2 g = \frac{1}{8\pi \sqrt{\lambda}} \int e^{\pm i \sqrt{|\lambda|} |y-x|} g(y) dy. \]

From these relations we obtain immediately the estimate, valid for all \( \lambda \in \mathbb{R} \) and \( \varepsilon \geq 0 \) with \( (\lambda, \varepsilon) \neq (0,0) \)
\[ \| R_0(\lambda \pm i\varepsilon)^2 g \|_{L^\infty} \leq \frac{1}{8\pi \sqrt{\lambda}} \| g \|_{L^1}. \]

3. The perturbed operator

Properties of Schrödinger operators with a potential in the Kato class are well known, see e.g. [22], [10], [25]. Under assumption (1.7), and actually even under much weaker assumptions, one can prove that \( H = -\Delta + V \) defines a self-adjoint nonnegative operator in \( L^2 \). For convenience of the reader, we sketch the proof in the following

**Lemma 3.1.** Let \( V = V_+ - V_- \) with \( V_+ \geq 0 \) be a measurable function on \( \mathbb{R}^3 \) satisfying
\[ V_+ \text{ is of Kato class, } \| V_- \|_K < 4\pi. \]

Then there exists a unique nonnegative self-adjoint operator \( H = -\Delta + V \) with domain \( \mathcal{D}(H) = H^2(\mathbb{R}^3) \) such that
\[ (\varphi, H\psi)_{L^2} = (\varphi, -\Delta\psi)_{L^2} + (\varphi, V\psi)_{L^2} \geq 0 \quad \forall \varphi, \psi \in H^2(\mathbb{R}^3). \]
Proof. We shall use the KLMN Theorem (see [22], Vol. II, Theorem 10.17). Thus it is sufficient to verify the following inequality:

\[(3.3) \quad \int_{\mathbb{R}^3} |V(x)||\varphi(x)|^2 dx \leq a \int_{\mathbb{R}^3} |\nabla \varphi(x)|^2 dx + b\|\varphi\|_{L^2(\mathbb{R}^3)}^2 \]

for some constants \(a < 1\), \(b \in \mathbb{R}\) and for all test functions \(\varphi\) (whence the same inequality is true for all \(\varphi \in H^1\) which is the domain of the form \(- (\Delta \varphi, \varphi)\)).

First of all we prove that for some \(a \in [0, 1]\) and for all \(b > 0\)

\[(3.4) \quad \int_{\mathbb{R}^3} V_-(x)|\varphi(x)|^2 dx \leq a \int_{\mathbb{R}^3} |\nabla \varphi(x)|^2 dx + b\|\varphi\|_{L^2(\mathbb{R}^3)}^2 \]

This can be written equivalently

\[|(V_- \varphi, \varphi)_{L^2}| \leq a(\varphi, -\Delta \varphi)_{L^2} + b\|\varphi\|_{L^2}^2 = a \left\| \left( H_0 + \frac{b}{a} \right)^{\frac{1}{2}} \varphi \right\|_{L^2}^2, \]

where \(H_0 = -\Delta\) is the selfadjoint operator with domain \(H^2(\mathbb{R}^3)\). Thus, writing \(g = (H_0 + \frac{b}{a})^{\frac{1}{2}} \varphi\), we see that we need only to prove the following inequality:

\[\left\| \left| V_- \right|^{\frac{1}{2}} \left( H_0 + \frac{b}{a} \right)^{-\frac{1}{2}} g \right\|_{L^2} \leq a\|g\|_{L^2}, \]

for some \(1 > a > 0\) and all \(b > 0\).

If we introduce the operator \(T = |V_-|^{\frac{1}{2}} \left( H_0 + \frac{b}{a} \right)^{-\frac{1}{2}}\) and its adjoint

\[T^* = \left( H_0 + \frac{b}{a} \right)^{-\frac{1}{2}} |V_-|^{\frac{1}{2}}, \]

we must prove that

\[(3.5) \quad \|TT^*\|_{L^2} = a^2 < 1. \]

Recalling the explicit representation of resolvent in \(\mathbb{R}^3\):

\[\left( H_0 + \frac{b}{a} \right)^{-1} \varphi = \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-\sqrt{2\pi}|x-y|} \varphi(y) dy \]

we can write

\[\|TT^*\varphi\|_{L^2}^2 = \left\| |V_-|^{\frac{1}{2}} \left( H_0 + \frac{b}{a} \right)^{-1} |V_-|^{\frac{1}{2}} \varphi \right\|_{L^2}^2 = \frac{1}{(4\pi)^2} \int |V_-(x)| \left( \int e^{-\sqrt{2\pi}|x-y|} |V_-(y)|dy \right) \left( \int e^{-\sqrt{2\pi}|x-y|} |\varphi(y)|^2 dy \right) dx \]

and by the Cauchy-Schwartz inequality

\[\leq \frac{1}{(4\pi)^2} \int |V_-(x)| \left( \int \frac{e^{-\sqrt{2\pi}|x-y|}}{|x-y|} |V_-(y)|dy \right) \left( \int \frac{e^{-\sqrt{2\pi}|x-y|}}{|x-y|} |\varphi(y)|^2 dy \right) dx \]

\[= \frac{1}{(4\pi)^2} \int \left( \int \frac{e^{-\sqrt{2\pi}|x-y|}}{|x-y|} |V_-(x)|dx \right) \left( \int |V_-(y)|dy \right) \left( \int \frac{e^{-\sqrt{2\pi}|x-y|}}{|x-y|} |\varphi(y)|^2 dy \right) dx \]

Now by definition of Kato norm we have (for all \(x\) and any \(a, b > 0\))

\[(3.6) \quad \int \frac{e^{-\sqrt{2\pi}|x-y|}}{|x-y|} |V_-(x)|dx \leq \int \frac{|V_-(x)|}{|x-y|} dx \leq \|V_\|_{K} \]

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which implies
\begin{equation}
\|TT^*\|_{L^2} \leq \frac{\|V\|_K}{4\pi} \equiv a^2 < 1
\end{equation}
by assumption (3.1), and this proves (3.4).
To conclude the proof it is sufficient to show that for all test functions \( \varphi \), for all \( a > 0 \) and for some \( b = b(a) \in \mathbb{R} \),
\begin{equation}
\int_{\mathbb{R}^3} V_+(x)|\varphi(x)|^2dx \leq a \int_{\mathbb{R}^3} |\nabla \varphi(x)|^2dx + b\|\varphi\|_{L^2}^2
\end{equation}
The proof is almost identical to the above one; the only difference appears in estimate (3.6) where we split the integral as follows
\[
\int e^{-\sqrt{\pi}|x-y|}/|x-y|\ |V_+(y)|dy = \int_{|x-y|<r} + \int_{|x-y|\geq r}
\]
for arbitrary \( r > 0 \). Fix now \( \delta > 0 \); if we choose \( r > 0 \) small enough, the first integral can be made smaller than \( \delta \) by assumption (3.1); on the other hand, with \( r \) chosen, the second integral can be made smaller than \( \delta \) by choosing \( \delta \) large enough. In conclusion we have
\[
\int e^{-\sqrt{\pi}|x-y|}/|x-y|\ |V_+(y)|dy \leq 2\delta
\]
provided \( b \) in (3.8) is large enough.

Inequality (3.3) is now a trivial consequence of (3.4) and (3.8); thus the assumptions of the KLMN theorem are satisfied and we can construct \( H = -\Delta + V \) as a self-adjoint operator on \( H^2 \). To see that it is positive, we can write
\[
((-\Delta + V)\varphi, \varphi)_{L^2} = (-\Delta \varphi, \varphi)_{L^2} + (V\varphi, \varphi)_{L^2} \geq \|\nabla \varphi\|_{L^2}^2 - |(V\varphi, \varphi)_{L^2}|;
\]
by inequality (3.4) we have
\[
\geq (1-a)\|\nabla \varphi\|_{L^2}^2 - b\|\varphi\|_{L^2}^2 \geq -b\|\varphi\|_{L^2}^2
\]
for every \( b > 0 \), and this implies
\begin{equation}
((-\Delta + V)\varphi, \varphi)_{L^2} \geq 0.
\end{equation}

4. Spectral calculus for the perturbed operator

Lemma 3.1 gives us the possibility to apply the spectral theorem and hence to use the functional calculus for \( H = -\Delta + V \), i.e., given any function \( \phi(\lambda) \) continuous and bounded on \( \mathbb{R} \), we can define the operator \( \phi(H) \) on \( L^2 \) as
\begin{equation}
\phi(H)f = L^2 - \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int \phi(\lambda)[RV(\lambda + i\varepsilon) - RV(\lambda - i\varepsilon)]f d\lambda
\end{equation}
where
\[
RV(z) = (-\Delta + V - z)^{-1}
\]
is the resolvent operator for \( H \) (see the standard reference books, e.g., Proposition 1.9 in Vol. 2 of [23]). To use the notation \( RV(\lambda \pm i0) \) would require to give a meaning to such operators, i.e., to prove the limiting absorption principle for the perturbed operator. Actually this is not necessary in view of the dispersive estimate which is our goal here, and we shall follow a different strategy.

For \( z \) outside the positive real axis we have the well known identities
\begin{equation}
R_0(z) = (I + R_0(z)V) RV(z) = RV(z) (I + V R_0(z)).
\end{equation}
In order to invert and obtain alternative representations of \( RV \) in terms of \( R_0 \) we need the following proposition, which is the crucial result of the paper:
Proposition 4.1. Under the assumptions of Theorem 1.1 there exists \( \varepsilon_0 > 0 \) such that the bounded operators \( I + R_0(\lambda \pm i\varepsilon)V : L^\infty \to L^\infty \) are invertible for all \( \lambda \in \mathbb{R} \), \( 0 \leq \varepsilon \leq \varepsilon_0 \) with uniform bound

\[
\| (I + R_0(\lambda \pm i\varepsilon)V)^{-1} \|_{L^\infty \to L^\infty} \leq C \quad \text{for all } \lambda \in \mathbb{R}, \quad 0 \leq \varepsilon \leq \varepsilon_0.
\]

We need a few lemmas. The first one is a well known property of the free resolvent, see e.g. [1] or Vol.II of [11]:

Lemma 4.2. For all \( \lambda > 0 \) and \( \varepsilon \geq 0 \), the operators \( R_0(\lambda \pm i\varepsilon) \) are bounded from the weighted \( L^2(\langle x \rangle^2 dx) \) to the weighted \( L^2(\langle x \rangle^{-s} dx) \) spaces for any \( s > 1/2 \), and moreover

\[
\| \langle x \rangle^{-s} R_0(\lambda \pm i\varepsilon)f \|_{L^2} \leq \frac{C}{\sqrt{\lambda}} \| \langle x \rangle^s f \|_{L^2}
\]

The second lemma is sligthly modified from [22]:

Lemma 4.3. If \( V(x) \) is a compactly supported function in the Kato class, then there exists a sequence of functions \( V_\varepsilon \in C_0^\infty(\mathbb{R}^3) \) such that \( \| V_\varepsilon - V \|_\kappa \to 0 \) and \( \text{supp } V_\varepsilon \downarrow \text{supp } V \) as \( \varepsilon \to 0 \).

Proof. Notice that \( V \) has a finite Kato norm since it is compactly supported. Moreover, \( V \) belongs to \( L^1 \). Consider now a sequence of nonnegative radial mollifiers, i.e., let \( \rho(x) \in C_0^\infty(\mathbb{R}^3) \) be a nonnegative radial function with support in the ball \( \{ |x| \leq 1 \} \) such that \( \int \rho(x) dx = 1 \), and set \( \rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon) \). Then we have the following standard properties of the Newton potential \( 1/|x| \):

\[
\frac{1}{|x|} \ast \rho_\varepsilon \equiv \frac{1}{|x|} \quad \text{for } |x| \geq \varepsilon,
\]

\[
\frac{1}{|x|} \ast \rho_\varepsilon \leq \frac{1}{|x|} \quad \text{for all } |x| \neq 0.
\]

Define now \( V_\varepsilon = V \ast \rho_\varepsilon \); for fixed \( x \) we have

\[
\left| \int \frac{V_\varepsilon(z)}{|x - z|} dz \right| \leq \int \frac{1}{|x - y|} \left( \frac{1}{|x - y|} - \int \frac{\rho(z - y)}{|y - z|} dz \right) dy
\]

and by (4.6) the term in brackets is positive,

\[
\leq \int \frac{V(y)}{|x - y|} \left( \frac{1}{|x - y|} - \int \frac{\rho(z - y)}{|y - z|} dz \right) dy \leq \int_{|y| \leq \varepsilon} \frac{|V(y)|}{|x - y|} dy
\]

where in the last step we used (4.5). Taking the supremum in \( x \) and recalling Definition 1.5 we obtain that \( \| V_\varepsilon - V \| \to 0 \). Clearly, the support of \( V_\varepsilon \) is the set of points at distance \( \leq \varepsilon \) from the support of \( V \).

The following elementary property of Kato class functions was stated in the Introduction and we prove it here for sake of completeness:

Lemma 4.4. A compactly supported function of Kato class has a finite Kato norm.

Proof. We consider here the case \( n = 3 \) only, for general dimension the modifications are evident. Let \( V(x) \) be of Kato class with support contained in a ball \( B(0, R) \). Then by definition we have the uniform bound

\[
\int_{|x - y| \leq 1} |V(y)| dy \leq \int_{|x - y| \leq 1} \frac{|V(y)|}{|x - y|} dy \leq C_0
\]

for some \( C_0 \) independent of \( x \), and covering the support of \( V \) with a finite number of balls of radius 1 we obtain that \( V \in L^1 \). Thus we can write

\[
\int \frac{|V(y)|}{|x - y|} dy \leq \int_{|x - y| \leq 1} \frac{|V(y)|}{|x - y|} dy + \int_{|x - y| \geq 1} \frac{|V(y)|}{|x - y|} dy \leq C_0 + \| V \|_{L^1}
\]
which concludes the proof.

The next lemma is a property of the squared operator \((R_0V)^2\):

**Lemma 4.5.** Let \(V\) be a compactly supported function in the Kato class. Then for all \(\lambda > 0\), \(\varepsilon \geq 0\) and \(\delta > 0\) there exists a constant \(C_0\) depending only on \(\delta\) such that

\[
\|R_0(\lambda \pm i\varepsilon)VR_0(\lambda \pm i\varepsilon)Vf\|_{L^\infty} \leq \left( \delta + \frac{C_0}{\sqrt{\lambda}} \right) \|f\|_{L^\infty}.
\]

*Proof.* By the maximum principle, since \(R_0(z)\) is holomorphic in \(z\), it is sufficient to prove the estimate for \(\varepsilon = 0\), i.e., for the operators \(R_0(\lambda \pm i0)\). Let \(V_\varepsilon\) be given by Lemma 4.3; we can write

\[
R_0(\lambda \pm i0)V R_0(\lambda \pm i0)V = R_0(V - V_\varepsilon)V R_0(V - V_\varepsilon) + R_0 V_\varepsilon R_0 V_\varepsilon
\]

and using estimate (2.8) we obtain

\[
\|R_0VR_0Vf\|_{L^\infty} \leq (2\pi)^{-1}\|V\|_K \cdot \|V - V_\varepsilon\|_K \cdot \|f\|_{L^\infty} + \|R_0 V_\varepsilon R_0 V_\varepsilon f\|_{L^\infty}.
\]

Since the first term can be made less than \(2\delta\|f\|_{L^\infty}\) by letting \(\varepsilon \to 0\), we can fix \(\varepsilon = \varepsilon(\delta)\) chosen in the appropriate way and we see that it is sufficient to prove (4.7) with \(V\) replaced by \(V_\varepsilon\). Now we have

\[
|R_0 V_\varepsilon R_0 V_\varepsilon f(x)| \leq \int_{|x-y|<r} \frac{|V_\varepsilon|}{|x-y|} dy \|R_0 V_\varepsilon f\|_{L^\infty} + \int_{|x-y|\geq r} \frac{|V_\varepsilon|}{|x-y|} dy \|V_\varepsilon R_0 V_\varepsilon f\|_{L^\infty};
\]

the first term clearly satisfies

\[
\int_{|x-y|<r} \frac{|V_\varepsilon|}{|x-y|} dy \leq C \int_{|x-y|<r} \frac{dy}{|x-y|} = \sigma(r) \to 0
\]

since \(V_\varepsilon\) is bounded, so that we find

\[
|R_0 V_\varepsilon R_0 V_\varepsilon f(x)| \leq \sigma(r)\|V\|_K \|f\|_{L^\infty} + \frac{1}{r} \|V_\varepsilon R_\varepsilon V_\varepsilon f\|_{L^1},
\]

where in the last step we used the property

\[
\int \frac{|V_\varepsilon|}{|x-y|} dy \leq \int \frac{|V|}{|x-y|} dy
\]

already used in the course of the proof of Lemma 4.3. To estimate the second term in (4.9) we write for some \(s > 1/2\)

\[
\|V_\varepsilon R_\varepsilon V_\varepsilon f\|_{L^1} \leq \|\langle x \rangle^s V_\varepsilon\|_{L^2} \|\langle x \rangle^{-s} R_\varepsilon V_\varepsilon f\|_{L^2}
\]

and applying Lemma 4.2 we get

\[
\leq \frac{C}{\sqrt{\lambda}}\|\langle x \rangle^s V_\varepsilon\|_{L^2}^2 \|f\|_{L^\infty}.
\]

Finally, reverting to (4.9), we obtain

\[
|R_0 V_\varepsilon R_\varepsilon V_\varepsilon f(x)| \leq \left( \sigma(r)\|V\|_K \|f\|_{L^\infty} + \frac{1}{r} \|\langle x \rangle^s V_\varepsilon\|_{L^2}^2 C \frac{1}{\sqrt{\lambda}} \right) \|f\|_{L^\infty}
\]

Since \(V_\varepsilon \in C_0^\infty\), the proof of the lemma is concluded.

We prove now a fundamental compactness property:

**Lemma 4.6.** Let \(V\) be a compactly supported function in the Kato class. Then for all \(\lambda \in \mathbb{R}\), \(\varepsilon \geq 0\) the operator \(R_0(\lambda \pm i\varepsilon)V:\ L^\infty \to L^\infty\) and the operator \(VR_0(\lambda \pm i\varepsilon):\ L^1 \to L^1\) are compact operators. Moreover, if \(f \in L^\infty\) then the function \(R_0(\lambda \pm i\varepsilon)Vf\) belongs to the weighted space \(L^2(\langle x \rangle^{-1} dx)\) for all \(s > 1/2\) and \(\lambda, \varepsilon \geq 0\).
Proof. It is sufficient to prove the result in the case when $V$ is a smooth function with compact support. Indeed, by Lemma 4.3 $V$ can be approximated in the Kato norm by such functions $V^\varepsilon$; if we know that $R_0V^\varepsilon$ are compact operators, we can regard $R_0V$ as the limit of this sequence of operators in the $L(L^\infty;L^\infty)$ norm since

$$\|R_0V^\varepsilon - R_0V\|_{L(L^\infty;L^\infty)} \leq \frac{1}{4\pi}\|V^\varepsilon - V\|_K$$

and this implies that $R_0V$ is also compact. A similar argument holds for $VR_0$. Thus from now on we shall assume that $V$ is smooth.

If the support of $V$ is contained in the ball $\{|x| \leq M\}$, we see that, for all $|x| > 2M$ and $y$ in the support of $V$, we have $|x - y| \geq |x| - M \geq |x|/2$. Thus by the explicit representation of $R_0$ we get

$$|R_0Vf(x)| \leq \frac{1}{|x|} \int |V(y)f(y)| \, dy \leq \frac{2}{|x|} \int |Vf(y)| \, dy \leq \frac{2}{|x|}\|V\|_L^1\|f\|_{L^\infty} \quad \text{for } |x| \geq 2M.$$  

On the other hand, if $f_j$ is a bounded sequence in $L^\infty$, by the Ascoli-Arzelà theorem it is easy to show that the sequence $R_0Vf_j$ is precompact on any bounded set in $\mathbb{R}^3$. Using this compactness property for small $x$ and the above inequality for large $x$, by a diagonal procedure we obtain that $R_0Vf_j$ has a convergent subsequence on the whole $\mathbb{R}^3$.

To prove the compactness of $VR_0$ we write it as $VR_0 = A_r + B_r$ where

$$A_r g(x) = \frac{V(x)}{4\pi} \int \frac{e^{\pm i\sqrt{\varepsilon}|x-y|}}{|x-y|} e^{-\varepsilon|x-y|/2\sqrt{\varepsilon}} \chi_r(x-y)g(y) \, dy$$

and

$$B_r g(x) = \frac{V(x)}{4\pi} \int \frac{e^{\pm i\sqrt{\varepsilon}|x-y|}}{|x-y|} e^{-\varepsilon|x-y|/2\sqrt{\varepsilon}} (1 - \chi_r(x-y))g(y) \, dy;$$

here $\chi_r(y) = \chi(y/r)$ is a cutoff function equal to 1 for $x$ near the origin and vanishing for large $x$. It is easy to show that $B_r$ is a compact operator on $L^1$; indeed, it is a bounded operator from $L^1$ to $W^{1,1}(\Omega)$ for any bounded open set containing the support of $V$, while $W^{1,1}(\Omega)$ is compactly embedded in $L^1(\mathbb{R}^3)$ by the Rellich-Kondrachov Theorem. Since $\|A_r\|_{L(L^1;L^1)} \to 0$ as $r \to 0$, we regard as above $VR_0$ as the uniform limit of compact operators, and this concludes the proof.

The final statement can be proved again by using the fact that

$$|R_0Vf(x)| \leq \frac{2}{|x|}\|V\|_L^1\|f\|_{L^\infty} \quad \text{for } |x| \geq 2M$$

which of course is valid for any (even nonsmooth) function $V$ of Kato class, provided its support is contained in the ball $\{|x| \leq M\}$. Notice that $V \in L^1$ under these assumptions. Using this inequality for $|x| \geq 2M$, and the usual estimate

$$|R_0Vf(x)| \leq \frac{\|V\|_K}{4\pi} \|f\|_{L^\infty}$$

for $|x| \leq 2M$, we obtain easily that $R_0Vf \in L^2((x)^{-s}dx)$ for any $s > 1/2$. \[ \square \]

The above proof might be used to show that the range of $R_0V$ is contained in the space $C_0$ of continuous functions vanishing at infinity, however we shall not need this property here.

Everything is prepared for the proof of the proposition:

Proof. (of Proposition 4.1). From Lemma 2.5 it is clear that $\|R_0V\|_{L(L^\infty;L^\infty)} \to 0$ as $\lambda \to -\infty$, uniformly in $\varepsilon$. Hence $I + R_0V$ can be inverted by expansion in Neumann series for any $\lambda < -M$ provided $M > 0$ is large enough, and the inverse operator has a bound dependent only on $M$ (and $V$).
We now consider the case \( \lambda > -M, \varepsilon \geq 0 \). Let \( V_R \) be as in assumption (1.8) and write for brevity

\[
V_1 = V_R, \quad V_2 = V - V_R,
\]

and

\[
T = R_0(\lambda \pm i \varepsilon)V_1, \quad S = R_0(\lambda \pm i \varepsilon)V_2;
\]

our purpose is to invert \( I + T + S \) for all \( \lambda > -M, \varepsilon \geq 0 \) and find a uniform bound for the inverse operator.

We first invert \( I + S \): since by (2.8) the norm of \( S: L^\infty \to L^\infty \) is bounded by \( \|V_2\|_K/(4\pi) \), which is strictly smaller than 1 by assumption (1.8), the result follows again by a straightforward Neumann series expansion. We thus get, for all \( \lambda \in \mathbb{R}, \varepsilon \geq 0 \),

\[
\|(I + S)^{-1}\|_{L(L^\infty;L^\infty)} \leq (1 - \|V_2\|_K/(4\pi))^{-1}.
\]

(4.12)

We then invert \( I + T \) for large \( \lambda \). Lemma 4.5 ensures that \( \|T^2\|_{L(L^\infty;L^\infty)} \to 0 \) as \( \lambda \to \infty \). This implies that for any \( \delta \in [0, 1] \) we can find \( \lambda_\delta \) such that for all \( \lambda \geq \lambda_\delta \), \( I - T^2 \) is invertible with norm

\[
\|(I - T^2)^{-1}\|_{L(L^\infty;L^\infty)} \leq \frac{1}{1 - \delta}.
\]

(4.13)

Since \( I - T \) has norm in \( L(L^\infty;L^\infty) \) bounded by \( 1 + (4\pi)^{-1}\|V_1\|_K \) independently of \( \lambda \) and

\[
(I - T)(I - T^2)^{-1} = (I + T)^{-1},
\]

we conclude that also \( I + T \) is invertible for any \( \lambda \geq \lambda_\delta \) and \( \varepsilon \geq 0 \), with bound

\[
\|(I + T)^{-1}\|_{L(L^\infty;L^\infty)} \leq \frac{1}{1 - \delta}(1 + \|V_1\|_K/(4\pi)).
\]

(4.14)

Consider now for \( \lambda \geq \lambda_\delta \) and \( \varepsilon \geq 0 \) the operator

\[
S(I + T)^{-1};
\]

by the usual bound \( \|S\|_{L(L^\infty;L^\infty)} \leq \|V_2\|_K/(4\pi) \) and by (4.14) we obtain

\[
\|S(I + T)^{-1}\|_{L(L^\infty;L^\infty)} \leq \frac{1}{4\pi}\|V_2\|_K \frac{1}{1 - \delta} \left( 1 + \frac{\|V_1\|}{4\pi} \right) = \frac{\alpha}{1 - \delta},
\]

where the constant \( \alpha \), recalling the main assumption (1.8), satisfies

\[
\alpha \equiv \frac{1}{4\pi}\|V_2\|_K \left( 1 + \frac{\|V_1\|}{4\pi} \right) < 1.
\]

Thus we see that

\[
\|S(I + T)^{-1}\|_{L(L^\infty;L^\infty)} \leq \frac{\alpha}{1 - \delta} < 1
\]

provided \( \delta < 1 - \alpha \), i.e., provided \( \lambda_\delta \) is large enough. Thus, choosing a suitable value of \( \lambda_\delta \), we have that for \( \lambda \geq \lambda_\delta \) and \( \varepsilon \geq 0 \) the operator

\[
I + S(I + T)^{-1}
\]

is invertible. Finally, writing for \( \lambda \geq \lambda_\delta \) and \( \varepsilon \geq 0 \)

\[
(I + S + T)^{-1} = (I + T)^{-1}(I + S(I + T)^{-1})^{-1},
\]

we see that \( I + S + T = I + R_0V \) is invertible with the bound

\[
\|(I + S + T)^{-1}\|_{L(L^\infty;L^\infty)} \leq \left( 1 + \frac{\|V_1\|}{4\pi} \right) \frac{1}{1 - \alpha - \delta}.
\]

(4.15)

It remains to invert \( I + S + T \) for \( -M \leq \lambda \leq \lambda_\delta \) with a uniform bound. Notice that it is sufficient to prove that for each \( \lambda \in [-M, \lambda_\delta] \) and \( \varepsilon_0 \geq \varepsilon \geq -\varepsilon_0 \) a bounded
inverse exists; then a bound independent of \(\lambda, \varepsilon\) will automatically follow from the continuity of the operation \(A \mapsto (I + A)^{-1}\). To this end we write formally

\[(I + S + T)^{-1} = (I + S)^{-1}(I + T(I + S)^{-1})^{-1};\]

if we show that the right hand side is well defined and bounded, this will prove that also the inverse operator \((I + S + T)^{-1}\) exists and is bounded, with the same bound.

Choose now \(\varepsilon_0\) so small that the rectangle \([-M, \lambda_0] \times [-\varepsilon_0, \varepsilon_0]\) is contained in the neighbourhood given by assumption (1.9). By Lemma 4.6 the operator \(T\) is compact, and hence the same holds for \(T(I + S)^{-1}\). By Fredholm theory we know that \(I + T(I + S)^{-1}\) is invertible if and only if it is one to one. Thus suppose by contradiction that \((I + T(I + S)^{-1})f = 0\) for some \(L^\infty\) function \(f\); this implies, writing

\[g = (I + S)^{-1} f\]

(notice that \(g \in L^\infty\))

\[g = -(T + S)g = -R_0 V g \in L^2((x)^{-s}dx)\]

for all \(s > 1/2\), by the last part of Lemma 4.6. If we now apply \((-\lambda \mp \iota \varepsilon - \Delta)\) to both sides we obtain

\[(-\lambda \mp \iota \varepsilon - \Delta + V)g = 0, \quad g \in L^2((x)^{-s}dx).\]

In other words \(\lambda\) is a resonance for \(-\Delta + V\) and this contradicts assumption (1.9) unless \(f \equiv g \equiv 0\). We deduce that \(I + T(I + S)^{-1}\) is one to one and, by Fredholm theory, invertible. Finally, \(I + T(I + S)^{-1}\) is invertible for all \(\lambda > -M, \varepsilon \geq 0\) and the inverse has a \(L(L^\infty; L^\infty)\) norm bounded by a constant independent of both parameters. Recalling (4.16), we see that the proof of the proposition is concluded.

We can now draw some consequences which are important for the following steps.

**Corollary 4.7.** Under the assumptions of Theorem 1.1 there exists \(\varepsilon_0 > 0\) such that the bounded operators \(I + VR_0(\lambda \pm \iota \varepsilon) : L^1 \to L^1\) are invertible for all \(\lambda \in \mathbb{R}, 0 \leq \varepsilon \leq \varepsilon_0\) with uniform bound

\[(I + VR_0(\lambda \pm \iota \varepsilon))^{-1} \in L(L^1; L^1), \quad \|I + VR_0(\lambda \pm \iota \varepsilon))^{-1}\|_{L(L^1; L^1)} \leq C\]

for all \(\lambda \in \mathbb{R}, 0 \leq \varepsilon \leq \varepsilon_0\).

**Proof.** The operators \(I + VR_0\) are one to one on \(L^1\) by duality, since by Proposition 4.1 the operators \(I + R_0 V\) are onto. They are onto by Fredholm theory, since \(VR_0\) are compact operators on \(L^1\) by Lemma 4.6. Finally, the bound on the inverse also follows by duality and the bound (4.3); indeed, \((L^1)' = L^\infty\) and hence

\[\|(I + VR_0)f\|_{L^1} = \sup_{\|h\|_{L^\infty} = 1} \int h(I + V R_0) f dx = \sup_{\|h\|_{L^\infty} = 1} \int f(I + R_0 V) h dx.\]

A consequence of (4.18) and of the proposition are the following representation formulas:

\[(4.18) \quad R_V(z) = (I + R_0 V)^{-1} R_0(z) = R_0(z) (I + VR_0)^{-1}\]

which hold for any \(z\) outside the positive real axis. By combining these relations we easily obtain the identity

\[(4.19) \quad R_V(\lambda + \iota \varepsilon) - R_V(\lambda - \iota \varepsilon) =
= (I + R_0(\lambda - \iota \varepsilon) V)^{-1}(R_0(\lambda + \iota \varepsilon) - R_0(\lambda - \iota \varepsilon)) (I + VR_0(\lambda + \iota \varepsilon))^{-1}\]
for all $\lambda \in \mathbb{R}$, $\varepsilon \in [0, \varepsilon_0]$. Then by the bounds (2.7) and (4.3), (4.17) we obtain
\begin{equation}
\| [R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)] g \|_{L^\infty} \leq C\sqrt{\lambda \varepsilon} \| g \|_{L^1}.
\end{equation}
for all $\lambda \in \mathbb{R}$, $\varepsilon \in [0, \varepsilon_0]$.

Moreover from (4.18) we get
\begin{equation}
R_V(\lambda \pm i\varepsilon)^2 = (I + R_0(\lambda \pm i\varepsilon)V)^{-1}R_0(\lambda \pm i\varepsilon)^2(I + VR_0(\lambda \pm i\varepsilon))^{-1}
\end{equation}
and recalling (2.14) we obtain
\begin{equation}
\| R_V(\lambda \pm i\varepsilon)^2 g \|_{L^\infty} \leq C\sqrt{\lambda \varepsilon} \| g \|_{L^1}.
\end{equation}
for all $\lambda \in \mathbb{R}$, $\varepsilon \in [0, \varepsilon_0]$.

The above results could be used to define in a suitable sense the limit operators $R_V(\lambda \pm i0)$, i.e., to prove that the limit absorption principle holds also for the perturbed operator, but this would lead us too far from the proof of the dispersive estimate (1.10).

5. Equivalence of Besov norms

This section is devoted to prove the equivalence of perturbed and standard Besov spaces
\begin{equation}
\dot{B}^s_{1,q}(\mathbb{R}^3) \sim \dot{B}^s_{1,q}(V)
\end{equation}
which holds for $s < 2$ under our assumptions. An analogous property holds for non homogeneous spaces.

We begin by adapting to our situation a result of Simon [22] (whose proof we follow closely). We believe that estimates (5.4) and (5.6) are of independent interest, thus we shall give the $n$-dimensional version. If the negative part of the potential is in the Kato class but not small, by Theorem B.1.1 of [22] the semigroup is still bounded, but its norm may increase exponentially as $t \to \infty$.

**Proposition 5.1.** Assume the potential $V = V_+ - V_-$ on $\mathbb{R}^n$, $n \geq 3$, $V_\pm \geq 0$, satisfies
\begin{equation}
V_+ \text{ is of Kato class}
\end{equation}
and
\begin{equation}
\| V_- \|_K < c_n \equiv 2\pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right)
\end{equation}
and consider the selfadjoint operator $H = -\Delta + V$. Then for all $t > 0$ and $1 \leq p \leq q \leq \infty$ the semigroup $e^{-tH}$ is bounded from $L^p$ to $L^q$ with norm
\begin{equation}
\| e^{-tH} \|_{L^p(\mathbb{R}^n; L^q)} \leq \frac{(2\pi t)^{-\gamma}}{(1 - \| V_- \|_K/c_n)^2}, \quad \gamma = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q}\right).
\end{equation}
Moreover, under the stronger assumption
\begin{equation}
\| V_- \|_K < \frac{1}{2}c_n
\end{equation}
e$^{-tH}$ is an integral operator with kernel $k(t, x, y)$ satisfying
\begin{equation}
|k(t, x, y)| \leq \frac{(2\pi t)^{-n/2}}{1 - 2\| V_- \|_K/c_n} e^{-|x - y|^2/8t}.
\end{equation}

**Proof.** In the following we shall use the more convenient notations
\begin{equation}
H = -\frac{1}{2}\Delta + V, \quad H_0 = -\frac{1}{2}\Delta;
\end{equation}
thus in the final step it will be necessary to substitute $t \to 2t$ and $V \to V/2$ in order to obtain the correct estimates.
The fundamental tool will be the Feynman-Kac formula

\[(e^{-tH} f)(x) = E_x \left( \exp \left( - \int_0^t V(b(s)) ds \right) f(b(t)) \right) \]

which is valid under much more general assumptions (see e.g. [25]). Here \(E_x\) is the integral over the path space \(\Omega\) with respect to the Wiener measure \(\mu_x, x \in \mathbb{R}^n\), while \(b(t)\) represents a generic path (brownian motion). We shall not need the full power of the theory but only a few basic facts:

i) Given a non negative function \(G(x)\) on \(\mathbb{R}^n\) we have the identity

\[E_x \left( \int_0^t G(b(s)) ds \right) = \int Q_t(x - y) G(y) dy \]

where \(Q_t(x)\) is the function

\[Q_t(x) = \int_0^t (2\pi s)^{-n/2} e^{-|x|^2/2s} ds. \]

It is easy to see by rescaling that

\[\int_0^\infty (2\pi s)^{-n/2} e^{-|x|^2/2s} ds = \int_0^\infty \tau^{-n/2} e^{-\tau |x|^2} \frac{1}{2\pi^{n/2}} = \Gamma \left( \frac{n}{2} - 1 \right) \frac{|x|^{2-n}}{2\pi^{n/2}} \]

so that by definition of \(c_n\) (see (5.3))

\[Q_t(x) \leq \frac{1}{c_n |x|^{n-2}} \]

and by (5.9)

\[E_x \left( \int_0^t G(b(s)) ds \right) \leq \frac{1}{c_n} \|G\|_K. \]

ii) Khasminskii’s lemma ([15]; B.1.2 in [22]): if \(G(x)\) is a non negative function on \(\mathbb{R}^n\) such that for some \(t\)

\[\alpha \equiv \sup_x E_x \left( \int_0^t G(b(s)) ds \right) < 1, \]

then

\[\sup_x E_x \left( \exp \left( \int_0^t G(b(s)) ds \right) \right) \leq \frac{1}{1 - \alpha}. \]

An immediate application is the following: if \(V_-\) satisfies

\[\|V_-\|_K < c_n \]

we have

\[\alpha \equiv \sup_x E_x \left( \int_0^t V_-(b(s)) ds \right) \leq \frac{1}{c_n} \|V_\|_K < 1 \]

by (5.12), so that

\[\sup_x E_x \left( \exp \left( \int_0^t V_-(b(s)) ds \right) \right) \leq \frac{1}{1 - \|V_\|_K/c_n}. \]

These simple facts gives us the first \(L^\infty - L^\infty\) estimate for the semigroup. Indeed, by the Feynman-Kač formula we have

\[\|e^{-tH} f\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} E_x \left( \exp \left( - \int_0^t V(b(s)) ds \right) f(b(t)) \right) \leq \|f\|_{L^\infty} E_x \left( \exp \left( - \int_0^t |V_-(b(s))| ds \right) \right) \leq \frac{\|f\|_{L^\infty}}{1 - \|V_\|_K/c_n}. \]
The second step is a $L^2 - L^\infty$ estimate. By the Feynman-Kač formula and the Schwarz inequality

\begin{equation}
|e^{-tH}f(x)| \leq E_x \left( \exp \left( -2 \int_0^t V_-(b(s))ds \right) \right)^{1/2} E_x \left( \|f(b(t))\|_1 \right)^{1/2} \equiv \left[ (e^{-t(\mathcal{H}_0+2V)})_1(x) \right]^{1/2} [e^{-t\mathcal{H}_0}\|f\|_2]^{1/2}
\end{equation}

where in the last step we used again the formula; now $e^{-t\mathcal{H}_0}$ is the standard heat kernel which has norm $(2\pi t)^{-n/2}$ as an $L^1 - L^\infty$ operator, while we can apply estimate (5.16) to the operator $e^{-t(\mathcal{H}_0+2V)}$. We thus obtain

\begin{equation}
|e^{-tH}f(x)| \leq \frac{\|1\|_{L^\infty}}{1 - 2\|V_-\|_K/c_n} (2\pi t)^{-n/4}\|f\|_{L^2}
\end{equation}

which implies

\begin{equation}
\|e^{-tH}f\|_{L^\infty} \leq \frac{(2\pi t)^{-n/4}}{1 - 2\|V_-\|_K/c_n} \|f\|_{L^2},
\end{equation}

provided

\[ \|V_-\|_K < \frac{c_n}{2}. \]

By duality, since $e^{-tH}$ is selfadjoint, we obtain the $L^2 - L^\infty$ estimate

\begin{equation}
\|e^{-tH}f\|_{L^2} \leq \frac{(2\pi t)^{-n/4}}{1 - 2\|V_-\|_K/c_n} \|f\|_{L^1},
\end{equation}

using the semigroup property we can write

\[ e^{-tH}f = e^{-\frac{t}{2}H}e^{-\frac{t}{2}H}f \]

and applying (5.18) first, then (5.19) we obtain

\begin{equation}
\|e^{-tH}f\|_{L^\infty} \leq \frac{(\pi t)^{-n/2}}{1 - 2\|V_-\|_K/c_n^2} \|f\|_{L^1}.
\end{equation}

Now recalling (5.16), by duality and interpolation we obtain

\[ \|e^{-tH}f\|_{L^p} \leq \frac{(\pi t)^{-n/2}}{1 - 2\|V_-\|_K/c_n^2} \|f\|_{L^3} \]

(the constant could be slightly but not essentially improved) with $\gamma$ as in the statement. The change $t \to 2t$, $V \to V/2$ gives (5.4).

Let now $g(x), h(x)$ be bounded functions; the same argument as in (5.17) gives

\[ |e^{-tH}h(x)| \leq \left[ (e^{-t(\mathcal{H}_0+2V)}|h|)(x) \right]^{1/2} \left[ e^{-t\mathcal{H}_0}|h|(x) \right]^{1/2} \]

and multiplying by $g(x)$ and taking the sup we get

\begin{equation}
\|ge^{-tH}h\|_{L^\infty} \leq \|ge^{-t(\mathcal{H}_0+2V)}|h|\|_{L^\infty} \|ge^{-t\mathcal{H}_0}|h|\|_{L^\infty}^{1/2}.
\end{equation}

We choose

\[ g = \chi_{K_1}, \quad h = f\chi_{K_2} \]

where $f(x)$ is a bounded function while $\chi_{K_1}, \chi_{K_2}$ are the characteristic functions of two disjoint compact sets $K_1, K_2$. We may estimate the first factor in (5.21) using (5.20) as follows

\[ \|ge^{-t(\mathcal{H}_0+2V)}|h|\|_{L^\infty} \leq \|e^{-t(\mathcal{H}_0+2V)}|h|\|_{L^\infty} \leq \frac{(\pi t)^{-n/2}}{1 - 4\|V_-\|_K/c_n^2} \|f\chi_{K_2}\|_{L^1} \]

while for the second we may use the explicit kernel of $e^{-t\mathcal{H}_0}$ i.e.,

\[ (2\pi t)^{-n/2} \exp(-|x - y|^2/2t) \]
and we obtain
\[ \|g e^{-tH_0}|h\|_{L^\infty} \leq (2\pi)^{-n/2} \exp(-d^2/2t) \|f\chi_{K_2}\|_{L^1}, \quad d = \text{dist}(K_1, K_2). \]

In conclusion we have
\[ (5.22) \quad \|\chi_{K_1} e^{-tH} f\chi_{K_2}\|_{L^\infty} \leq \left(\frac{\pi t}{1 - 4\|V_\|_{K}/c_n}\right)^{n/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \|f\chi_{K_2}\|_{L^1}, \quad d = \text{dist}(K_1, K_2). \]

By the Dunford-Pettis Theorem (see Trèves [24] and A.1.1-A.1.2 in [22]), this implies at once that \( e^{-tH} \) has an integral kernel representation, with kernel
\[ k(t, x, y) = \frac{(\pi t)^{-n/2}}{1 - 4\|V_\|_{K}/c_n} e^{-|x-y|^2/4t} \]
and this concludes the proof \((t \rightarrow 2t, V \rightarrow V/2)\). \(\square\)

We shall now use the above kernel representation of the semigroup to improve a result due to Jensen and Nakamura (Theorem 2.1 in [12]):

**Proposition 5.2.** Assume the Kato class potential \( V = V_+ - V_- \) on \( \mathbb{R}^n, n \geq 3 \), \( V_\geq 0 \), satisfies
\[ (5.23) \quad \|V_+\|_K < \infty \]
and
\[ (5.24) \quad \|V_-\|_K < \frac{1}{2} c_n \equiv \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} - 1\right)} \]
and consider the selfadjoint operator \( H = -\Delta + V \). Then for any \( g \in C_0^\infty(\mathbb{R}) \) and any \( \theta > 0 \) the operator \( g(\theta H) \) is bounded on \( L^p(\mathbb{R}^n), 1 \leq p \leq \infty \), with norm independent of \( \theta \):
\[ (5.25) \quad \|g(\theta H)\|_{L(L^p, L^p)} \leq C(p, n, g, V). \]
The same property holds for the rescaled operators
\[ (5.26) \quad \|g(H_\theta)\|_{L(L^p, L^p)} \leq C(p, n, g, V), \]
where \( H_\theta = -\Delta + \theta V(\sqrt{\theta}x) \).

**Proof.** The proof for fixed \( \theta \) is quite lengthy and is contained in [13]. In [12], Theorem 2.1, the result was extended to a uniform estimate like (5.25) for \( 0 < \theta \leq 1 \), under assumptions on the potential weaker than ours. Following that proof, in order to extend the result to \( \theta \geq 1 \) it will be sufficient to prove that a few estimates are uniform in \( \theta \geq 1 \). More precisely, consider the rescaled potential
\[ (5.27) \quad V_\theta(x) = \theta V(\sqrt{\theta}x); \]
notice that the Kato norm is invariant under this transformation:
\[ (5.28) \quad \|V_\theta\|_K \equiv \|V\|_K. \]
Then consider the operator
\[ (5.29) \quad H_\theta = -\Delta + V_\theta; \]
following [12], it is sufficient to prove that
\[ (5.30) \quad \|g(H_\theta)\|_{L(L^p, L^p)} \leq C \]
uniformly in \( \theta \), and this amounts to proving three estimates:
   i) a uniform pointwise estimate for the kernel of \( e^{-tH_\theta} \),
   ii) an \( L^2 - L^2 \) estimate for the operator \( (H_\theta + M)^{-1/2}, M > 0 \) a fixed constant (we can take \( M = 1 \) here),
   iii) an \( L^2 - L^2 \) estimate for the operator \( \partial_x (H_\theta + M)^{-1/2} \).
Step i) follows directly from estimate (5.6)

\[(5.31)\quad |k_0(t, x, y)| \leq \frac{(2\pi t)^{-n/2}}{1-2\|V_\theta-\|_K/c_n} e^{-|x-y|^2/4t}.\]

which is uniform in \(\theta > 0\) since by (5.27)

\[\|V_\theta-\|_K \equiv \|V_-\|_K\]
does not depend on \(\theta\).

Step ii) is trivial since \(\|(H_\theta + M)^{-1/2}\|_{L(L^2,L^2)} \leq M^{-1/2}\). To get iii), we must prove that

\[\|\partial_x (H_\theta + M)^{-1/2} f\|_{L^2} \leq C\|f\|_{L^2}\]
or equivalently

\[(5.32)\quad \|g\|_{H^1} \leq C\|(H_\theta + M)^{1/2} g\|_{L^2}\]

for some \(C\) independent of \(\theta > 0\). We rewrite (5.32) as

\[(5.33)\quad C^{-1}\|g\|_{H^1} \leq (-\Delta g, g) + (V_\theta g, g) + M\|g\|_{L^2}^2.\]

Clearly (5.33) is implied by

\[(5.34)\quad |(V_\theta-g, g)| \leq \alpha \|g\|_{H^1} + M\|g\|_{L^2}^2, \quad \alpha < 1, \quad \alpha \text{ independent of } \theta.\]

Now recall (3.4), where we proved the inequality in dimension \(n = 3\): for all \(b > 0\)

\[(5.35)\quad |(V_2 \varphi, \varphi)| \leq a(-\Delta \varphi, \varphi) + b\|\varphi\|_{L^2}\]

where by (3.7)

\[(5.36)\quad a^2 = \frac{\|V_2\|_K}{4\pi}.\]

We can now apply (5.35), (5.36) to \(V_\theta-\) whose Kato norm is independent of \(\theta\):

\[a^2 = \frac{\|V_\theta-\|_K}{4\pi} = \frac{\|V_-\|_K}{4\pi} < \frac{c_3}{8\pi} = \frac{1}{4}\]

by (5.24), and this concludes the proof of iii) in dimension \(n = 3\).

The proof for \(n \geq 3\) is identical; it is sufficient to modify the proof of (3.4), (3.7) using the \(n\)-dimensional representation

\[(5.37)\quad (H_\theta + M)^{-1} f(x) = \int_0^\infty \int (2\pi s)^{-n/2} e^{-|x-y|^2/2s} e^{-sM} f(y) dy\ dx\]

(recall also (5.10)-(5.11)). We omit the straightforward details.

We now prepare a useful lemma:

**Corollary 5.3.** Assume \(V\) satisfies the assumptions of Proposition 5.2, let \(H_\theta = -\Delta + \theta V(\sqrt{\theta}x), H_0 = -\Delta,\) and let \(\varphi_j(s) = \varphi_0(2^{-j}s), \psi_j(s) = \psi_0(2^{-j}s)\) be two homogeneous Paley-Littlewood partitions of unity, \(j \in \mathbb{Z}\). Then we have the estimates: for all \(j, k \in \mathbb{Z}\),

\[(5.38)\quad \|\varphi_j(\sqrt{H_\theta})\psi_k(\sqrt{H_0})\|_{L(L^1,L^1)} \leq C 2^{-2j+2k}\]

with a constant \(C\) independent of \(j, k\) and of \(\theta > 0\). The same estimates hold interchanging \(H_0\) and \(H_\theta\).

**Proof.** We first note two consequences of (5.25): for all \(j\), with a constant independent of \(j\),

\[(5.39)\quad \|\varphi_j(\sqrt{H_\theta}) H_{0}\|_{L(L^1,L^1)} \leq C 2^{2j}, \quad \|\varphi_j(\sqrt{H_\theta}) H_{0}^{-1}\|_{L(L^\infty;L^2)} \leq C 2^{-2j}\]

and the analogous ones for \(H_0\) instead of \(H\) (indeed, the case \(V = 0\) is a special case of (5.39)). The first one follows by choosing

\[g(s) = \varphi_0(\sqrt{s})s \implies g(2^{-2j}H_\theta) = \varphi_j(\sqrt{H_\theta})2^{-2j}H_\theta;\]
the second one follows by
\[ g(s) = \varphi_0(\sqrt{s})s^{-1} \implies g(2^{-2j}H_\theta) = \varphi_j(\sqrt{H_\theta})2^{2j}H_\theta^{-1}. \]
Then we can write
\[ \varphi_j(\sqrt{H_\theta})\psi_k(\sqrt{H_\theta}) = \varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_\theta\psi_k(\sqrt{H_\theta}) = \varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_0\psi_k(\sqrt{H_\theta}) + \varphi_j(\sqrt{H_\theta})H_\theta^{-1}V_\theta\psi_k(\sqrt{H_\theta}). \]
The first term can be estimated immediately using (5.39):
\[ \|\varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_0\psi_k(\sqrt{H_\theta})\|_{L(L^p,L^p)} \leq C2^{-2j+2k}; \]
for the second one we may write
\[ \|\varphi_j(\sqrt{H_\theta})H_\theta^{-1}V_\theta\psi_k(\sqrt{H_\theta})\|_{L(L^p,L^p)} \leq C2^{-2j}\|V_\theta\psi_k(\sqrt{H_\theta})\|_{L(L^p,L^p)} \]
and since
\[ V_\theta\psi_k(\sqrt{H_\theta}) = V_\theta R_0(0)H_0\psi_k(\sqrt{H_\theta}), \]
recalling that \( V_\theta R_0 \) is a bounded operator on \( L^1 \) (with norm proportional to the Kato norm of \( V_\theta \) which does not depend on \( \theta \)) and applying again (5.39) we obtain (5.38).

We omit the straightforward details for higher dimension \( n > 3 \). The only modification consists in showing that also in higher dimensions the operator \( R_0(0)V \) is bounded in \( L^\infty \), provided \( V \) is in the Kato class. This follows easily by an estimate of the kernel of \( R_0(0) \) like
\[ |K(x,y)| \leq \frac{c}{|x-y|^{n-2}} \]
which is standard (see e.g. [19]). \( \Box \)

Using Corollary 5.3 we can show the equivalence of non homogeneous Besov spaces \( B^s_{p,q}(V) \) with the standard ones, and later on we shall prove the more difficult result concerning the homogeneous case. We recall the precise definition: given a homogeneous Paley-Littlewood partition of unity \( \varphi_j(s) = \varphi_0(2^{-j}s), j \in \mathbb{Z}, \) we set for \( p \in [1,\infty], q \in [1,\infty], s \in \mathbb{R} \)
\[ \|f\|_{B^s_{p,q}(V)} = \left( \sum_{j \in \mathbb{Z}} 2^{jsq}\|\varphi_j(\sqrt{H})f\|_{L^p}^q \right)^{1/q} \]
with obvious modification when \( q = \infty \). When \( V = 0 \) we obtain the classical Besov spaces, and we write simply
\[ \|f\|_{B^s_{p,q}} \text{ instead of } \|f\|_{B^s_{p,q}(0)}. \]

On the other hand, if we consider a non homogeneous Paley-Littlewood partition of unity, i.e., \( \varphi_j \) as above for \( j \geq 0 \), and we set
\[ \psi_0 = 1 - \sum_{j \geq 0} \varphi_j \]
clearly we obtain \( \psi_0 \in C_0^\infty(\mathbb{R}^n) \) and we can define the non homogeneous Besov norm as
\[ \|f\|_{B^s_{p,q}(V)} = \left( \|\psi_0(\sqrt{H})f\|_{L^p}^q + \sum_{j \geq 0} 2^{jsq}\|\varphi_j(\sqrt{H})f\|_{L^p}^q \right)^{1/q} \]
We write simply \( B^s_{p,q} \) in the standard case \( V = 0 \).
Theorem 5.4. Assume the Kato class potential $V = V_+ - V_-$ on $\mathbb{R}^n$, $n \geq 3$, $V_+ \geq 0$, satisfies
\begin{equation}
\|V_+\|_K < \infty
\end{equation}
and
\begin{equation}
\|V_-\|_K < \frac{1}{2}c_n \equiv \frac{\pi^{n/2}}{\Gamma \left(\frac{n}{2} - 1\right)}
\end{equation}
Then we have the equivalence of norms
\begin{equation}
\|f\|_{B^s_{1,q}(V)} \sim \|f\|_{B^s_{1,q}(V_0)}
\end{equation}
for all $q \in [1, \infty]$, $0 \leq s < 2$. Moreover, for the rescaled potentials
\begin{equation}
V_\theta(x) = \theta V(\sqrt{\theta} x)
\end{equation}
we have the uniform estimates
\begin{equation}
C^{-1}\|f\|_{B^s_{1,q}} \leq \|f\|_{B^s_{1,q}(V_\theta)} \leq C\|f\|_{B^s_{1,q}}
\end{equation}
with a constant $C$ independent of $\theta > 0$.

Remark 5.1. In order to improve the result and consider higher values of $s \geq 2$ stronger smoothness assumptions on the of the potential $V$ are necessary; we shall not pursue this problem here. Also, to prove the equivalence of Besov spaces $B^s_{p,q}$ for $p \neq 1$, one should prove different bounds for the operator $V R_0$ on $L^p$; this is possible but quite technical and we limit ourselves to the case $p = 1$ which is our main interest here.

Proof. We shall limit ourselves to the case $q = 1$ and we shall only prove the inequality
\begin{equation}
\|f\|_{B^s_{1,1}(V_\theta)} \leq C\|f\|_{B^s_{1,1}};
\end{equation}
the proof of the reverse inequality and of the cases $1 < q \leq \infty$ are completely analogous.

In the following we shall drop the index $\theta$ since all the estimates we use (from Proposition 5.2 and Corollary 5.3) have constants independent of $\theta > 0$.

Using the notations
\begin{equation}
D_V = \sqrt{H}, \quad D = \sqrt{H_0}
\end{equation}
we have
\begin{equation}
\|f\|_{B^s_{1,1}(V)} = \|\psi_0(D_V) f\|_{L^1} + \sum_{j=0}^{\infty} 2^{js}\|\phi_j(D_V) f\|_{L^1}.
\end{equation}
Using
\begin{equation}
1 = \psi_0(D) + \sum_{k \geq 0} \varphi_k(D),
\end{equation}
we have
\begin{align*}
\|f\|_{B^s_{1,1}(V)} & \leq \|\psi_0(D_V)\psi_0(D) f\|_{L^1} + \sum_{k=0}^{\infty}\|\psi_0(D_V)\varphi_k(D) f\|_{L^1} + \\
& + \sum_{j=0}^{\infty} 2^{js}\|\phi_j(D_V)\psi_0(D) f\|_{L^1} + \sum_{j,k \geq 0} 2^{js}\|\phi_j(D_V)\varphi_k(D) f\|_{L^1} = I + II + III + IV.
\end{align*}
We estimate separately the four terms.

Since by (5.26) $\psi_0(D_V)$ is bounded on $L^1$, we have for the first term
\begin{equation}
I = \|\psi_0(D_V)\psi_0(D) f\|_{L^1} \leq C\|f\|_{L^1}
\end{equation}
and since
\[ \|f\|_{L^1} \leq \|\psi_0(D)f\|_{L^1} + \sum_{j \geq 0} \|\varphi_j(D)f\|_{L^1} \]
this is smaller than \(\|f\|_{B^{1}_{1,1}}\).

The same argument gives for the second term
\[ II = \sum_{k=0}^{\infty} \|\psi_0(DV)\varphi_k(D)f\|_{L^1} \leq C \sum_{k=0}^{\infty} \|\varphi_k(D)f\|_{L^1} \leq C\|f\|_{B^{1}_{1,1}} \]
As to the third term, we can write
\[ \sum_{j=0}^{\infty} 2^{js}\|\varphi_j(DV)\psi_0(D)f\|_{L^1} = \sum_{j=0}^{\infty} 2^{js}\|\varphi_j(DV)(-\Delta V)^{-1}(-\Delta V)\psi_0(D)f\|_{L^1} \]
and recalling (5.39) used in the proof of the corollary we have (for \(s < 2\))
\[ III \leq C \sum_{j \geq 0} 2^{-j(2-s)}\|(-\Delta V)\psi_0(D)f\|_{L^1} = C\|(-\Delta V)\psi_0(D)f\|_{L^1} \leq C\|(-\Delta)\psi_0(D)f\|_{L^1} + C\|V\psi_0(D)f\|_{L^1}. \]
Now we have
\[ \|V\psi_0(D)f\|_{L^1} = \|VR_0(0)(-\Delta)\psi_0(D)f\|_{L^1} \leq \frac{\|V\|_{C}}{4\pi} \|(-\Delta)\psi_0(D)f\|_{L^1} \]
and since \((-\Delta)\psi_0(D)\) is bounded in \(L^1\) by (5.26), we conclude that
\[ (5.48) \text{III} \leq C_2\|f\|_{L^1} \leq C_3\|f\|_{B^{1}_{1,1}} \]
as for the first term.

Finally, we split the fourth term in the two sums for \(j \leq k\) and \(j > k\): \[ IV = \sum_{j,k \geq 0} 2^{js}\|\varphi_j(DV)\varphi_k(D)f\|_{L^1} = \sum_{j \leq k} + \sum_{j > k}. \]
For \(j \leq k\) we use the fact that \(\varphi_j(DV)\) are bounded on \(L^1\) with uniform norm by (5.26) and hence
\[ \sum_{j \leq k} \leq C \sum_{k \geq 0} \|\varphi_k(D)f\|_{L^1} \sum_{0 \leq j \leq k} 2^{js} = 2C \sum_{k \geq 0} 2^{ks}\|\varphi_k(D)f\|_{L^1}. \]
For \(j > k\), we write \(\varphi_j = \varphi_j(\varphi_j+1) + \varphi_j(\varphi_j+1) = \varphi_j\widehat{\varphi_j}\) and have
\[ \sum_{j > k} 2^{js}\|\varphi_j(DV)\varphi_k(D)f\|_{L^1} = \sum_{j > k} 2^{js}\|\varphi_j(DV)\varphi_k(D)f\|_{L^1}; \]
now by the corollary we obtain
\[ \sum_{j > k} 2^{js}\|\varphi_j(DV)\varphi_k(D)f\|_{L^1} \leq \sum_{j > k} C2^{(k-j)(2-s)}2^{ks}\|\widehat{\varphi_k}f\|_{L^1}. \]
and since \(\sum_{j > k} 2^{(k-j)(2-s)} < 1\) we have
\[ (5.49) IV = \sum_{j,k \geq 0} 2^{js}\|\varphi_j(DV)\varphi_k(D)f\|_{L^1} \leq C \sum_{k \geq 0} 2^{k}\|\widehat{\varphi_k}f\|_{L^1} \leq C\|f\|_{B^{1}_{1,1}(R^q)}. \]
and this concludes the proof. \(\square\)

We shall finally show that the preceding result implies the equivalence also for homogeneous Besov spaces. Indeed, the uniformity of estimates (5.44) makes it possible to apply a rescaling argument, using the following lemma:
Lemma 5.5. Let $s \in \mathbb{R}$, $p, q, \in [1, \infty]$. The homogeneous $\dot{B}_{p,q}^s(V)$ norm has the following rescaling property with respect to scaling $(S_\lambda f)(x) = f(\lambda x)$:

\begin{equation}
\|S_\lambda f\|_{\dot{B}_{p,q}^s(V)} = \lambda^{-\frac{n}{p}} \|f\|_{\dot{B}_{p,q}^s(V_{\lambda^{-2}})}
\end{equation}

provided $\lambda = 2^k$ for some $k \in \mathbb{Z}$.

Remark 5.2. Of course the property holds also for any positive $\lambda$, with identity replaced by equivalence of norms; we shall not need this improvement here.

Proof. From the identity

$$(-\Delta + V(x))S_\lambda f(x) = \lambda^2 S_\lambda(-\Delta + \lambda^{-2}V(x/\lambda))f(x)$$

we obtain the rule

$$\Delta_V S_\lambda = \lambda^2 S_\lambda \Delta_{V_{\lambda^{-2}}}$$

with the usual notations

$$\Delta_V = \Delta + V, \quad V_\theta = \theta V(\sqrt{\theta}x).$$

This implies

$$g(-\Delta_V)S_\lambda = S_\lambda g(-\lambda^2 \Delta_{V_{\lambda^{-2}}})$$

and in particular for the functions $\phi_j(s) = \phi_0(2^{-j}s)$, writing as usual $D_V = \sqrt{-\Delta_V}$,

$$\phi_j(D_V)S_\lambda = \phi_0(2^{-j}D_V)S_\lambda = S_\lambda \phi_0(2^{-j}\lambda D_{V_{\lambda^{-2}}}).$$

With the special choice $\lambda = 2^k$ this can be written

$$\phi_j(D_V)S_{2^k} = S_{2^k} \phi_{j-k}(D_{V_{2^{-k}}}).$$

Hence we have the identity, for $\lambda = 2^k$,

$$\|S_\lambda\|_{\dot{B}_{p,q}^s}^q = \sum_{j \in \mathbb{Z}} 2^{jqs}\|\phi_j(D_V)S_\lambda f\|_{L^p}^q \leq \sum_{j \in \mathbb{Z}} 2^{jqs}2^{-knq/p}\|S_\lambda \phi_{j-k}(D_{V_{2^{-k}}})f\|_{L^p}^q$$

since $L^p$ rescales as $\lambda^{-n/p}$; writing $2^{jqs}2^{-knq/p} = 2^{k(s-n/p)q} 2^{(j+k)sq}q$ and shifting the sum $j + k \to j$ we conclude the proof. $\square$

Thus we arrive at the final result of this section:

Theorem 5.6. Assume the Kato class potential $V = V_+ - V_-$ on $\mathbb{R}^n$, $n \geq 3$, $V_{\pm} \geq 0$, satisfies

\begin{equation}
\|V_+\|_K < \infty
\end{equation}

and

\begin{equation}
\|V_-\|_K < \frac{1}{2} c_n \equiv \pi^{n/2}/\Gamma\left(\frac{n}{2} \right) \left(\frac{n}{2} - 1\right)
\end{equation}

Then we have the equivalence of norms

\begin{equation}
\|f\|_{\dot{B}_{p,q}^s(V_\theta)} \cong \|f\|_{\dot{B}_{1,q}^{s}}
\end{equation}

for all $q \in [1, \infty]$, $0 < s < 2$. Moreover, for the rescaled potentials

$$V_\theta(x) = \theta V(\sqrt{\theta}x)$$

we have the uniform estimates

\begin{equation}
C^{-1} \|f\|_{\dot{B}_{1,q}^{s}} \leq \|f\|_{\dot{B}_{1,q}^{s}(V_\theta)} \leq C \|f\|_{\dot{B}_{1,q}^{s}}
\end{equation}

with a constant $C$ independent of $\theta > 0$.
Proof. We shall consider in detail the case $q = 1$ only, the remaining cases being completely analogous.

We already know that (5.54) holds for dotless Besov spaces. Now we need the following inequalities:

\[(5.55) \quad C^{-1}\|f\|_{B_{1,1}^q(V_\theta)} \leq \|f\|_{B_{1,1}^q(V_\theta)} \leq C\|f\|_{B_{1,1}^q(V_\theta)} + C\|f\|_{B_{1,1}^q(V_\theta)}\]

which hold with a constant $C$ independent of $\theta > 0$.

First of all we prove that \((D = \sqrt{-\Delta}, D_{V_\theta} = \sqrt{-\Delta_{V_\theta}})\)

\[(5.56) \quad \sum_{j < -1} 2^j \|\varphi_j(D_{V_\theta})f\|_{L^1} \leq C\|\psi_0(D_{V_\theta})f\|_{L^1}.\]

We notice that $\psi_0$ is equal to 1 on the support of $\varphi_j$ for $j < -1$. Hence $\varphi_j = \varphi_j \psi_0$ for $j < -1$ and we can write

\[\|\varphi_j(D_{V_\theta})f\|_{L^1} = \|\varphi_j(D_{V_\theta})\psi_0(D_{V_\theta})f\|_{L^1} \leq C\|\psi_0(D_{V_\theta})f\|_{L^1}.\]

(we have used the uniform estimates (5.25)-(5.26)). Thus (5.56) follows, provided $s > 0$ so that $\sum_{j < -1} 2^j$ is convergent.

The term for $j = -1$ is estimated in a simple way \((\varphi_{-1} = \varphi_{-1}(\psi_0 + \varphi_1))\)

\[(5.57) \quad \|\varphi_{-1}(D_{V_\theta})f\|_{L^1} \leq \|\varphi_{-1}(D_{V_\theta})\psi_0(D_{V_\theta})f\|_{L^1} + \|\varphi_{-1}(D_{V_\theta})\varphi_1(D_{V_\theta})f\|_{L^1} \leq\]

\[\leq C\|\psi_0(D_{V_\theta})f\|_{L^1} + C\|\varphi_1(D_{V_\theta})f\|_{L^1}.\]

Clearly, (5.56) and (5.57) imply immediately the first inequality (5.55).

The second inequality in (5.55) is easier: it is sufficient to prove that

\[\|\psi_0(D_{V_\theta})f\|_{L^1} \leq C \sum_{j \leq 1} \|\varphi_j(D_{V_\theta})f\|_{L^1}\]

which follows from $\psi_0 = \sum_{j \leq 1} \varphi_j$, the triangle inequality, and the boundedness of $\psi_0(D_{V_\theta})$ on $L^1$ with uniform norm. Thus (5.55) is proved. Notice that all the constant appearing in the above inequalities are uniform in $\theta > 0$.

By (5.55) and the equivalence (5.44) we can write for $0 < s < 2

\[\|f\|_{B_{1,1}^q} \leq C\|f\|_{B_{1,1}^q} \leq C\|f\|_{B_{1,1}^q(V_\theta)} \leq C\|f\|_{B_{1,1}^q(V_\theta)} + C\|f\|_{B_{1,1}^q(V_\theta)}.\]

If we apply this inequality to a rescaled function $S_{2^k}f$ and recall Lemma 5.5, we obtain for all $k \in \mathbb{Z}$

\[2^{k(s-n)}\|f\|_{B_{1,1}^q} \leq C2^{k(s-n)}\|f\|_{B_{1,1}^q(V_{2^k} - 2^{k\lambda})} + C2^{k-n}\|f\|_{B_{1,1}^q(V_{2^k} - 2^{k\lambda})}\]

with constants independent of $k, \theta$; we can now choose $\theta = 2^{k\gamma}$, divide by $2^{k(s-n)}$ and let $k \to +\infty$ to obtain

\[\|f\|_{B_{1,1}^q} \leq C\|f\|_{B_{1,1}^q(V_\gamma)}\]

which is the first part of the thesis. The reverse inequality is proved in the same way. \(\square\)

6. Conclusion of the proof

By the spectral calculus for $H = -\Delta + V$, given any bounded continuous function $\phi(s)$ on $\mathbb{R}$, we can represent the operator $\phi(H)$ on $L^2$ as

\[(6.1) \quad \phi(H)f = L^2 - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int \phi(\lambda)[R_{V}(\lambda + i\varepsilon) - R_{V}(\lambda - i\varepsilon)]f d\lambda.\]

If $\phi = \psi'$ is the derivative of a $C^1$ compactly supported function we can integrate by parts obtaining the equivalent form

\[(6.2) \quad \phi(H)f = -L^2 - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int \psi(\lambda)[R_{V}(\lambda + i\varepsilon)^2 - R_{V}(\lambda - i\varepsilon)^2]f d\lambda.\]
Now, fix a smooth function $\psi(s)$ with compact support in $]0, +\infty[$ and consider the Cauchy problem
\begin{equation}
\{ \Box u + V(x)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^3 \\
u(0, t) = 0, \quad u_t(0, x) = \psi(H)g
\end{equation}
for some smooth $g$. Then the solution $u$ can be represented as
\begin{equation}
u(t, \cdot) = L^2 - \lim_{\varepsilon \to 0} u_\varepsilon(t, \cdot)
\end{equation}
where
\begin{equation}
u_\varepsilon(t, x) = \frac{1}{2\pi i} \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} \psi(\lambda)[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]g d\lambda
\end{equation}
or equivalently, after integration by parts,
\begin{equation}
u_\varepsilon(t, x) = \frac{1}{\pi it} \int_0^\infty \cos(t\sqrt{\lambda})\psi(\lambda)[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]g d\lambda + \frac{1}{\pi it} \int_0^\infty \cos(t\sqrt{\lambda})\psi(\lambda)[R_V(\lambda + i\varepsilon)^2 - R_V(\lambda - i\varepsilon)^2]g d\lambda.
\end{equation}
Estimates (4.20) and (4.22) applied to (6.5) give
\begin{equation}
\|u_\varepsilon(t, \cdot)\|_{L^\infty} \leq \|g\|_{L^1} \frac{C}{t} \int_0^\infty \left( |\psi'(\lambda)|\sqrt{\lambda} + \frac{|\psi(\lambda)|}{\sqrt{\lambda}} \right) d\lambda
\end{equation}
and recalling that
\begin{equation}
\lambda \leq \lambda_\varepsilon \leq \lambda + \frac{\varepsilon}{2}
\end{equation}
we obtain
\begin{equation}
\|u_\varepsilon(t, \cdot)\|_{L^\infty} \leq \|g\|_{L^1} \frac{C}{t} \int_0^\infty \left( |\psi'(\lambda)|(\sqrt{\lambda} + \sqrt{\varepsilon}) + \frac{|\psi(\lambda)|}{\sqrt{\lambda}} \right) d\lambda.
\end{equation}

Let now $\varphi_j(s)$, $j \in \mathbb{Z}$ be the homogeneous Paley-Littlewood partition of unity defined in the Introduction, with
\begin{equation}
\varphi_j(s) = \phi_0(2^{-j}s),
\end{equation}
define
\begin{equation}
\tilde{\varphi}_j(s) = \varphi_{j-1}(s) + \varphi_j(s) + \varphi_{j+1}(s)
\end{equation}
and choose in (6.3)
\begin{equation}
\psi(\lambda) = \tilde{\varphi}_j(\sqrt{\lambda}) \equiv \tilde{\varphi}_0(2^{-j}\sqrt{\lambda}).
\end{equation}
We thus obtain
\begin{equation}
\|u_\varepsilon(t, \cdot)\|_{L^\infty} \leq \|g\|_{L^1} \frac{C}{t} \int_0^\infty \left( 2^{-j}|\tilde{\varphi}_0(2^{-j}\sqrt{\lambda})|\frac{\sqrt{\lambda} + \sqrt{\varepsilon}}{2\sqrt{\lambda}} + \frac{|\tilde{\varphi}_0(2^{-j}\sqrt{\lambda})|}{\sqrt{\lambda}} \right) d\lambda
\end{equation}
which after the change of variables $\mu = 2^{-j}\sqrt{\lambda}$ gives
\begin{equation}
\|u_\varepsilon(t, \cdot)\|_{L^\infty} \leq \frac{C}{t} (2^j + \sqrt{\varepsilon})\|g\|_{L^1}.
\end{equation}
for some constant $C$ independent of $j$, $t$ and $g$. If we let $\varepsilon \to 0$, for fixed $t$ the functions $u_\varepsilon(t, \cdot)$ converge in $L^2$ to the solution $u(t, x)$; hence a subsequence converges a.e. and we obtain the estimate
\begin{equation}
\|u(t, \cdot)\|_{L^\infty} \leq \frac{C}{t} 2^j \|g\|_{L^1}
\end{equation}
for the solution $u(t, x)$ of the Cauchy problem
\begin{equation}
\{ \Box u + V(x)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^3 \\
u(0, t) = 0, \quad u_t(0, x) = \tilde{\varphi}_j(\sqrt{H})g
\end{equation}
If we now choose
\[ g = \varphi_j(\sqrt{H})f \]
and notice that \( \tilde{\varphi}_j g \equiv \tilde{\varphi}_j \varphi_j f \equiv \varphi_j f \) since \( \tilde{\varphi}_j = 1 \) on the support of \( \varphi_j \), we conclude that: the solution \( u(t, x) \) of the Cauchy problem
\[
\begin{cases}
\Box u + V(x)u = 0, & t \geq 0, \ x \in \mathbb{R}^3 \\
u(0, t) = 0, & u_t(0, x) = \varphi_j(\sqrt{H})f
\end{cases}
\]
satisfies the estimate
\[
\|u(t, \cdot)\|_{L^\infty} \leq C \frac{2^j}{t} \left\| \varphi_j(\sqrt{H})f \right\|_{L^1}.
\]
Consider now the original Cauchy problem (1.4); decomposing the initial datum \( f \) as
\[ f = \sum_{j \in \mathbb{Z}} \varphi_j(\sqrt{H})f \]
applying estimate (6.12) and summing over \( j \), we obtain by linearity that the solution \( u(t, x) \) to (1.4) satisfies the estimate
\[
\|u(t, \cdot)\|_{L^\infty} \leq C \frac{1}{t} \left\| f \right\|_{B^1_{1,1}(V)}.
\]
Since by Theorem 5.6 this norm is equivalent to the standard one, we see that the proof of Theorem 1.1 is concluded.

References

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