Simultaneous linearization of holomorphic germs in presence of resonances

JASMIN RAISSY
Dipartimento di Matematica, Università di Pisa
Largo Bruno Pontecorvo 5, 56127 Pisa
E-mail: raissy@mail.dm.unipi.it

ABSTRACT. Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $(df_1)_O$ diagonalizable and such that $f_1$ commutes with $f_h$ for any $h = 2, \ldots, m$. We prove that, under certain arithmetic conditions on the eigenvalues of $(df_1)_O$ and some restrictions on their resonances, $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_m$.

1. Introduction

One of the main questions in the study of local holomorphic dynamics (see [A] and [B] for general surveys on this topic) is when a given germ of biholomorphism $f$ of $\mathbb{C}^n$ at a fixed point $p$, which we may place at the origin $O$, is holomorphically linearizable, i.e., there exists a local holomorphic change of coordinates, tangent to the identity, conjugating $f$ to its linear part. The answer to this question depends on the set of eigenvalues of $df_O$, usually called the spectrum of $df_O$. In fact, if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ the eigenvalues of $df_O$, then it may happen that there exists a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $|k| := k_1 + \cdots + k_n \geq 2$ and such that

$$\lambda^k - \lambda_j := \lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0$$

for some $1 \leq j \leq n$; a relation of this kind is called a resonance of $f$, and $k$ is called a resonant multi-index. A resonant monomial is a monomial $z^k = z_1^{k_1} \cdots z_n^{k_n}$ in the $j$-th coordinate such that $\lambda^k = \lambda_j$.

One possible generalization of the previous question is to ask when a given set of $m \geq 2$ germs of biholomorphisms $f_1, \ldots, f_m$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are simultaneously holomorphically linearizable, i.e., there exists a local holomorphic change of coordinates conjugating $f_h$ to its linear part for each $h = 1, \ldots, m$.

In [R] we found, under certain arithmetic conditions on the eigenvalues and some restrictions on the resonances, a necessary and sufficient condition for holomorphic linearization. In this article we shall use that result to find a necessary and sufficient condition for holomorphic simultaneous linearization.

Before stating our result we need the following definitions:

**Definition 1.1.** Let $1 \leq s \leq n$. We say that $\lambda = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ has only level $s$ resonances if there are only two kinds of resonances:

(a) $\lambda^k = \lambda_h \iff k \in \tilde{K}_1,$

where

$$\tilde{K}_1 = \left\{ k \in \mathbb{N}^n \mid |k| \geq 2, \sum_{p=1}^{s} k_p = 1 \text{ and } \mu_1^{k_{s+1}} \cdots \mu_r^{k_n} = 1 \right\};$$
We say that $f$ commutes with the reduced Brjuno condition if there exists a strictly increasing sequence of integers $(\lambda_1, \ldots, \lambda_n)$ relative to $1$ (i.e., when there are no resonances) is nothing but the usual Brjuno condition introduced in [Br] (see also [M] pp. 25–37 for the one-dimensional case).

Definition 1.2. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For any $m \geq 2$ put

$$\tilde{\omega}(m) = \min_{2 \leq |k| \leq m} \min_{1 \leq j \leq n} |\lambda^k - \lambda_j|,$$

where $\text{Res}_j(\lambda)$ is the set of multi-indices $k \in \mathbb{N}^n$, with $|k| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., such that $\lambda^k - \lambda_j = 0$. We say that $\lambda$ satisfies the reduced Brjuno condition if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_{\nu}^{-1} \log \tilde{\omega}(p_{\nu+1})^{-1} < \infty.$$

Note that the reduced Brjuno condition of order $n$ (i.e., when there are no resonances) is nothing but the usual Brjuno condition introduced in [Br] (see also [M] pp. 25–37 for the one-dimensional case).

Definition 1.3. Let $f$ be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin $O$ and let $s \in \mathbb{N}$, with $1 \leq s \leq n$. The origin $O$ is called a quasi-Brjuno fixed point of order $s$ if $df_O$ is diagonalizable and, denoting by $\lambda$ the spectrum of $df_O$, we have:

(i) $\lambda$ has only level $s$ resonances;

(ii) $\lambda$ satisfies the reduced Brjuno condition.

We say that $f$ has the origin as a quasi-Brjuno fixed point if there exists $1 \leq s \leq n$ such that it is a quasi-Brjuno fixed point of order $s$.

Definition 1.4. Let $f_1, \ldots, f_m$ be $m$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let $M$ be a germ of complex manifold at $O$ of codimension $1 \leq s \leq n$, and $f_h$-invariant for each $h = 1, \ldots, m$. We say that $M$ is a simultaneous osculating manifold for $f_1, \ldots, f_m$ if there exists a holomorphic flat $(1,0)$-connection $\nabla$ of the normal bundle $N_M$ of $M$ in $\mathbb{C}^n$ commuting with $df_h|_{N_M}$ for each $h = 1, \ldots, m$.

In [R] we saw that the osculating condition was necessary and sufficient to extend a holomorphic linearization from an invariant submanifold to a whole neighbourhood of the origin for a germ $f_1$ of biholomorphism with a quasi-Brjuno fixed point. Our main theorem shows that the simultaneous osculating condition is also necessary and sufficient to extend a common holomorphic linearization, just assuming that $f_1$ has a quasi-Brjuno fixed point and commutes with $f_2, \ldots, f_m$:

Theorem 1.1. Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that $f_1$ has the origin as a quasi-Brjuno fixed point of order $s$, with $1 \leq s \leq n$, and
that it commutes with $f_h$ for any $h = 2, \ldots, m$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold $M$ at $O$ of codimension $s$, invariant under $f_h$ for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1^{\mid M}, \ldots, f_m^{\mid M}$ are simultaneously holomorphically linearizable.

A similar topic is studied in [S]. However, his results are not comparable with ours, because his notion of “linearization modulo an ideal” is not suitable for producing a full linearization result, except when there are no resonances at all, whereas in our result we explicitly admit some resonances.

We shall need the following notation: if $g: \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with $g(O) = 0$, and $z = (x,y) \in \mathbb{C}^n$ with $x \in \mathbb{C}^s$ and $y \in \mathbb{C}^{n-s}$, we shall denote by $\text{ord}_x(g)$ the maximum positive integer $m$ such that $g$ belongs to the ideal $(x_1, \ldots, x_s)^m$. Furthermore, we shall say that the local coordinates $z = (x,y)$ are adapted to the complex submanifold $M$ if in those coordinates $M$ is given by $\{x = 0\}$.

2. Linearization

We first introduced osculating manifolds in [R]. A germ $f$ of biholomorphism of $\mathbb{C}^n$ fixing the origin $O$ admits an osculating manifold $M$ of codimension $1 \leq s \leq n$ if there is a germ of $f$-invariant complex manifold $M$ at $O$ of codimension $s$ such that the normal bundle $N_M$ of $M$ admits a holomorphic flat $(1,0)$-connection that commutes with $df|_{N_M}$. Definition 1.4 is the natural extension of this object to the case we are dealing with.

We shall need the following characterization of simultaneous osculating manifolds.

**Proposition 2.1.** Let $f_1, \ldots, f_m$ be $m$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let $M$ be a germ of complex manifold at $O$ of codimension $1 \leq s \leq n$, and $f_h$-invariant for each $h = 1, \ldots, m$. Then $M$ is a simultaneous osculating manifold for $f_1, \ldots, f_m$ if and only if there exist local holomorphic coordinates $z = (x,y)$ about $O$ adapted to $M$ in which $f_h$ has the form

\[
x_i' = \sum_{p=1}^{s} a_{i,p}^{(h)} x_p + \hat{f}_i^{(h)}(x,y) \quad \text{for} \quad i = 1, \ldots, s,
\]

\[
y_j' = f_j^{(h)}(x,y) \quad \quad \quad \quad \quad \text{for} \quad j = 1, \ldots, r = n - s,
\]

with

\[\text{ord}_x(\hat{f}_i^{(h)}) \geq 2,\]

for any $i = 1, \ldots, s$ and $h = 1, \ldots, m$.

**Proof.** If there exist local holomorphic coordinates $z = (x,y)$ about $O$ adapted to $M$ in which $f_h$ has the form (2) with $\text{ord}_x(\hat{f}_i^{(h)}) \geq 2$ for any $i = 1, \ldots, s$ and $h = 1, \ldots, m$, then it is obvious to verify that the trivial holomorphic flat $(1,0)$-connection commutes with $df_h|_{N_M}$ for each $h = 1, \ldots, m$.

Conversely, let $\nabla$ be a holomorphic flat $(1,0)$-connection of the normal bundle $N_M$ commuting with $df_h|_{N_M}$ for each $h = 1, \ldots, m$. It suffices to choose local holomorphic coordinates $z = (x,y)$ adapted to $M$ in which all the connection coefficients $\Gamma_{jk}^{\ell}$ with respect to the local holomorphic frame $\{\pi(\frac{\partial}{\partial x_1}), \ldots, \pi(\frac{\partial}{\partial x_s})\}$ of $N_M$ are zero (see [R] Proposition 3.1 and Lemma 3.2), and then the assertion follows immediately from the proof of Theorem 1.3 of [R].

\[\square\]
Corollary 2.2. Let $f_1, \ldots, f_m$ be $m$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let $M$ be a germ of complex manifold at $O$ of codimension $1 \leq s \leq n$, and $f_h$-invariant for each $h = 1, \ldots, m$. Then $M$ is a simultaneous osculating manifold for $f_1, \ldots, f_m$ such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable if and only if there exist local holomorphic coordinates $z = (x, y)$ about $O$ adapted to $M$ in which $f_h$ has the form

$$
x_i' = \sum_{p=1}^s a_{i,p}^h x_h + {\tilde f}_i^{(h),1}(x, y) \quad \text{for } i = 1, \ldots, s,
$$

$$
y_j' = f_j^{(h)\text{lin}}(x, y) + {\tilde f}_j^{(h),2}(x, y) \quad \text{for } j = 1, \ldots, r = n - s,
$$

where $f_j^{(h)\text{lin}}(x, y)$ is linear and

$$
\text{ord}_x({\tilde f}_i^{(h),1}) \geq 2, \\
\text{ord}_x({\tilde f}_j^{(h),2}) \geq 1,
$$

for any $i = 1, \ldots, s, j = 1, \ldots, r$ and $h = 1, \ldots, m$.

Proof. One direction is clear.

Conversely, thanks to Proposition 2.1, the fact that $M$ is a simultaneous osculating manifold for $f_1, \ldots, f_m$ is equivalent to the existence of local holomorphic coordinates $z = (x, y)$ about $O$ adapted to $M$, in which $f_h$ has the form (3) with ord$_x(f_i^{(h),1}) \geq 2$ for any $i = 1, \ldots, s$ and $h = 1, \ldots, m$. Furthermore, $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable; therefore there exists a local holomorphic change of coordinate, tangent to the identity, and of the form

$$
\tilde x = x, \\
\tilde y = \Phi(y),
$$

conjugating $f_h$ to $\tilde f_h$ of the form (3) satisfying (4), for each $h = 1, \ldots, m$, as we wanted. \(\square\)

Remark 2.3. It is possible to give the formal analogous of Definition 1.4, and then to prove a formal analogous of Proposition 2.1 and Corollary 2.2, exactly as in [R].

In the proof of Theorem 1.1 we shall use the following result we proved in [R]

Theorem 2.4. (Raissy, 2007) Let $f$ be a germ of biholomorphism of $\mathbb{C}^n$ having the origin $O$ as a quasi-Brjuno fixed point of order $s$. Then $f$ is holomorphically linearizable if and only if it admits an osculating manifold $M$ of codimension $s$ such that $f|_M$ is holomorphically linearizable.

We can now prove our result.

Theorem 2.5. Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that $f_1$ has the origin as a quasi-Brjuno fixed point of order $s$, with $1 \leq s \leq n$, and that it commutes with $f_h$ for any $h = 2, \ldots, m$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold $M$ at $O$ of codimension $s$, invariant under $f_h$ for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable.

Proof. Let $M$ be a germ of complex manifold at $O$ of codimension $s$, invariant under $f_h$ for each $h = 1, \ldots, m$ which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such
that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable. Thanks to the hypotheses we can choose local holomorphic coordinates 

$$(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_r)$$

such that $f_1$ is of the form

$$x'_i = \lambda_{1,i}x_i + f_1^{(1),1}(x, y) \quad \text{for } i = 1, \ldots, s,$$
$$y'_j = \mu_{1,j}y_j + f_1^{(1),2}(x, y) \quad \text{for } j = 1, \ldots, r = n - s,$$

and, for $h = 2, \ldots, m$, each $f_h$ is of the form

$$x'_i = \sum_{p=1}^s a_{i,p}^{(h)} x_p + f_h^{(h),1}(x, y) \quad \text{for } i = 1, \ldots, s,$$
$$y'_j = f_h^{(h),2}(x, y) + f_h^{(1),2}(x, y) \quad \text{for } j = 1, \ldots, r = n - s,$$

where $f_h^{(h),2}(x, y)$ is linear, and for each $k = 1, \ldots, m$

$$\text{ord}_x(f_k^{(k),1}) \geq 2,$$
$$\text{ord}_x(f_k^{(k),2}) \geq 1,$$

that is

$$f_k^{(k),1}(x, y) = \sum_{|K| \geq 2, |K'| \geq 2} f_k^{(k),1} x^{K'} y^{K''} \quad \text{for } i = 1, \ldots, s,$$
$$f_k^{(k),2}(x, y) = \sum_{|K| \geq 2, |K'| \geq 1} f_k^{(k),2} x^{K'} y^{K''} \quad \text{for } j = 1, \ldots, r,$$

where $K = (K', K'') \in \mathbb{N}^s \times \mathbb{N}^r = \mathbb{N}^n$ and $|K| = \sum_{p=1}^n K_p$.

Thanks to Theorem 2.4 and its proof, we know that $f_1$ is holomorphically linearizable via a linearization $\psi$ of the form

$$x_i = u_i + \psi_1^i(u, v) \quad \text{for } i = 1, \ldots, s,$$
$$y_j = v_j + \psi_2^j(u, v) \quad \text{for } j = 1, \ldots, r,$$

where $(u, v) = (u_1, \ldots, u_s, v_1, \ldots, v_r)$ and

$$\text{ord}_u(\psi_1^i) \geq 2,$$
$$\text{ord}_u(\psi_2^j) \geq 1,$$

that is

$$\psi_1^i(u, v) = \sum_{|K| \geq 2, |K'| \geq 2} \psi_{K,i}^1 u^{K'} v^{K''} \quad \text{for } i = 1, \ldots, s,$$
$$\psi_2^j(u, v) = \sum_{|K| \geq 2, |K'| \geq 1} \psi_{K,j}^2 u^{K'} v^{K''} \quad \text{for } j = 1, \ldots, r.$$
Since $\psi^{-1} \circ f_1 \circ \psi = \text{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$ commutes with $\tilde{f}_h = \psi^{-1} \circ f_h \circ \psi$ for each $h = 2, \ldots, m$, and $(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$ has only level $s$ resonances, it is immediate to verify that $\tilde{f}_h$ has the form

$$u'_i = \sum_{p=1}^{s} a^{(h)}_{1,p} u_p + \sum_{1 \leq i, j \leq n, \lambda_1, i = \lambda_1, i} u_{l,j} \tilde{f}^{(h),1}_{l, i}(v) \quad \text{for } i = 1, \ldots, s,$$

$$v'_j = f^{(h),1}_{j}(u, v) + \tilde{f}^{(h),2}_{j}(v) \quad \text{for } j = 1, \ldots, r.$$  

Moreover, since $f_h \circ \psi = \psi \circ \tilde{f}_h$, we have

$$\sum_{p=1}^{s} a^{(h)}_{1,p} \sum_{|K| \geq 2} \psi_1^{K,p} u^{K'} v^{K''} + \sum_{|K| \geq 2} f^{(h),1}_{K,i} (u + \psi^1(u, v)) K' (v + \psi^2(u, v)) K''$$

$$= \sum_{1 \leq i, j \leq n} u_{l,j} \tilde{f}^{(h),1}_{l, i}(v)$$

(5)

$$+ \sum_{|K| \geq 2} \psi^{K,i}_i \left( \sum_{p=1}^{s} a^{(h)}_{1,p} u_p + \sum_{1 \leq i, j \leq n, \lambda_1, i = \lambda_1, i} u_{l,j} \tilde{f}^{(h),1}_{l, i}(v) \right) \times \left( \sum_{p=1}^{s} a^{(h)}_{s,p} u_p + \sum_{1 \leq i, j \leq n, \lambda_1, i = \lambda_1, s} u_{l,j} \tilde{f}^{(h),1}_{l, s}(v) \right) K_s$$

for $i = 1, \ldots, s$, and

$$\sum_{q=1}^{r} b^{(h)}_{j,q} \sum_{|K| \geq 2} \psi_2^{2,q} u^{K'} v^{K''} + \sum_{p=1}^{s} c^{(h)}_{j,p} \sum_{|K| \geq 2} \psi_1^{K,p} u^{K'} v^{K''}$$

$$+ \sum_{|K| \geq 2} f^{(h),2}_{K,j} (u + \psi^1(u, v)) K' (v + \psi^2(u, v)) K''$$

(6)

$$= \tilde{f}^{(h),2}_{j}(v)$$

$$+ \sum_{|K| \geq 2} \psi^{K,i}_i \left( \sum_{p=1}^{s} a^{(h)}_{1,p} u_p + \sum_{1 \leq i, j \leq n, \lambda_1, i = \lambda_1, i} u_{l,j} \tilde{f}^{(h),1}_{l, i}(v) \right) \times \left( \sum_{p=1}^{s} a^{(h)}_{s,p} u_p + \sum_{1 \leq i, j \leq n, \lambda_1, i = \lambda_1, s} u_{l,j} \tilde{f}^{(h),1}_{l, s}(v) \right) K_s$$

for $j = 1, \ldots, r$.

Now, it is not difficult to verify that there are no terms of the form $u^{K'} v^{K''}$ with $|K'| = 1$ in the left-hand side of (5), whereas in the right-hand side terms of this form are given only by the sum of the $u_{l,j} \tilde{f}^{(h),1}_{l, i}(v)$; therefore it must be

$$\tilde{f}^{(h),1}_{l, i}(v) \equiv 0,$$

for all pairs $l, i$. Similarly, there are no terms of the form $u^{K'} v^{K''}$ with $K' = 0$ in the left-hand side of (6), whereas, again, in the right-hand side terms of this form are given by $\tilde{f}^{(h),2}_{j}(v)$ only; so

$$\tilde{f}^{(h),2}_{j}(v) \equiv 0 \quad \text{for } j = 1, \ldots, r.$$
This proves that \( \tilde{f}_h \) is linear for every \( h = 2, \ldots, m \), that is \( \psi \) is a simultaneous holomorphic linearization for \( f_1, \ldots, f_m \).

The other direction is clear. In fact, if \( f_1 \) commutes with \( f_2, \ldots, f_m \) and \( f_1, \ldots, f_m \) are linear, then the eigenspace of \( f_1 \) relative to the eigenvalues \( \mu_1, \ldots, \mu_r \) is a simultaneous osculating manifold for \( f_1, \ldots, f_m \) (and \( f_1|_M, \ldots, f_m|_M \) are linear), where \( (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \) is the spectrum of \( f_1 \).

**Corollary 2.6.** Let \( f_1, \ldots, f_m \) be \( m \geq 2 \) germs of commuting biholomorphisms of \( \mathbb{C}^n \), fixing the origin. Assume that \( f_1 \) has the origin as a quasi-Brjuno fixed point of order \( s \), with \( 1 \leq s \leq n \). Then \( f_1, \ldots, f_m \) are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold \( M \) at \( O \) of codimension \( s \), invariant under \( f_h \) for each \( h = 1, \ldots, m \) which is a simultaneous osculating manifold for \( f_1, \ldots, f_m \) and such that \( f_1|_M, \ldots, f_m|_M \) are simultaneously holomorphically linearizable.

As a final corollary, taking \( s = n \) in Theorem 2.5, one gets

**Corollary 2.7.** Let \( f_1, \ldots, f_m \) be \( m \geq 2 \) germs of biholomorphisms of \( \mathbb{C}^n \), fixing the origin. Assume that \( f_1 \) has the origin as a Brjuno fixed point, and that it commutes with \( f_h \) for any \( h = 2, \ldots, m \). Then \( f_1, \ldots, f_m \) are simultaneously holomorphically linearizable.

**References**


