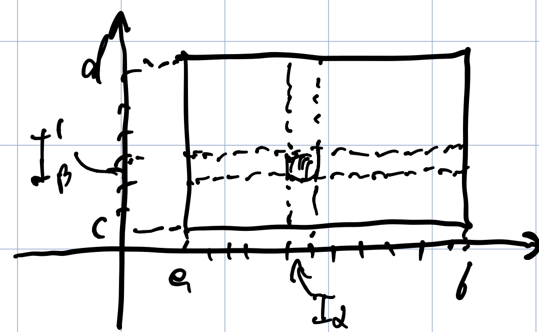
# Integrali multipli (doppi, tripli).

$f(x,y)$




$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$

$P_x = \{a = x_0 < x_1 < \dots < x_n = b\}$ $\qquad I_\alpha = [x_{\alpha-1}, x_\alpha]$

$P_y = \{c = y_0 < y_1 < \dots < y_h = d\}$ $\qquad I'_\beta = [y_{\beta-1}, y_\beta]$

$R$ è decomposto nei "rettangolini"

$$R_{\alpha\beta} = I_\alpha \times I_\beta = \{(x,y) \in \mathbb{R}^2 \mid x_{\alpha-1} \leq x \leq x_\alpha,\ y_{\beta-1} \leq y \leq y_\beta\}$$

somma di Riemann relativa alle partizioni $P_x, P_y$:

$$S(f,P) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{h} (\text{Area } R_{\alpha\beta}) \cdot f(P^*_{\alpha\beta})$$

$P^*_{\alpha\beta} \in R_{\alpha\beta}$

Def $f$ è integrabile su $R$ se le somme di Riemann tendono a un numero (finito) $I$ al tendere a zero dell'ampiezza della partizione.

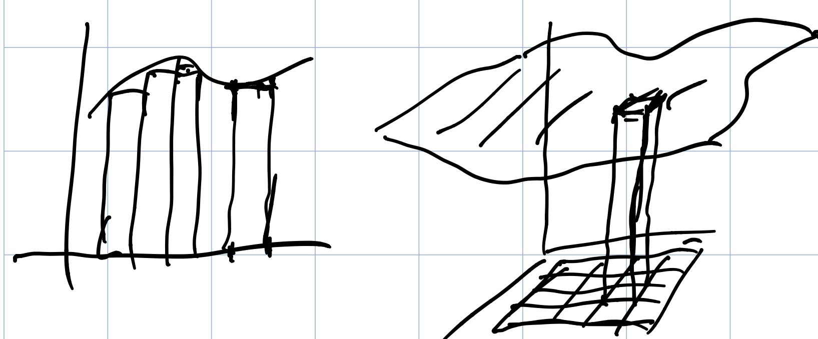
$$I = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

$\forall \varepsilon > 0 \ \exists \delta : \quad |S(f,P) - I| < \varepsilon$

quando $\quad |P| < \delta$

$|P|$ = ampiezza di $P$ = massimo dei diametri degli $R_{\alpha\beta}$.

(diametro: massima distanza fra due punti: nel caso di un rettangolo è la lunghezza della diagonale)



$S(f,P)$ approssima volume del sottografico (quando $f \geq 0$)

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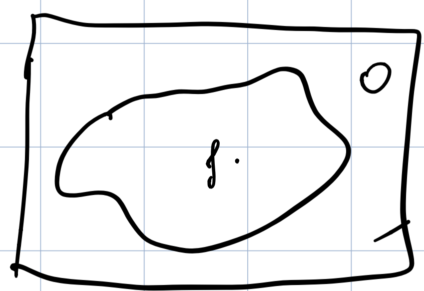
Per domini $D$ più generali, limitati,

prendiamo un rettangolo $R \supset D$ ed estendiamo $f$ a $D$ tramite $0$:

$$\hat{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \in R \setminus D \end{cases}$$

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**Def** $f(x,y)$ è integrabile in $D$ $\left( \iint_D f(x,y)\,dx\,dy \right)$ se $\hat{f}$ lo è in $R$.

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<u>Teorema</u> $f$ continua in un dominio $D$ chiuso e limitato, con bordo composto da un numero finito di curve di lunghezza finita $\Rightarrow$ $f$ integrabile in $D$

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- $\iint_D f(x,y)\,dA = 0$ se l'area di $D$ è nulla oss Se $D$ ha area nulla $\Rightarrow$ $D$ non ha punti interni

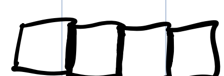
posso ricoprire $D$ con una unione di rettangoli (di cui so calcolare l'area) e l'inf delle aree di unione di quadrati che contengono $D$ è $0$

- $$\iint_D 1 \, dx\,dy = \text{Area } D$$

- $$\iint_D \left(\alpha f(x,y) + \beta g(x,y)\right) dx\,dy = \alpha \iint_D f(x,y)\,dx\,dy + \beta \iint_D g(x,y)\,dx\,dy$$

- $$\left| \iint_D f(x,y)\,dx\,dy \right| \leq \iint_D |f(x,y)|\,dx\,dy$$

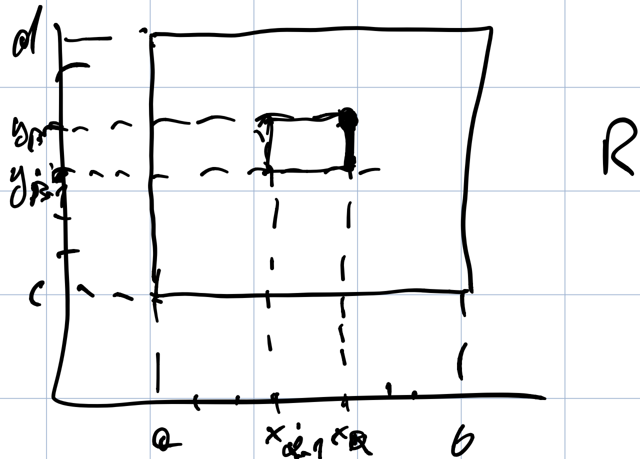
- $D = D_1 \cup \cdots \cup D_k$



se $D_i \cap D_j$ ha area nulla $\Rightarrow$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \cdots + \iint_{D_k} f(x,y)\,dx\,dy$$

METODO ITERATIVO.




$$S(f,\beta) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{h} \text{Area } R_{\alpha\beta} \cdot f(P^*_{\alpha\beta}) =$$

$$= \sum_{\alpha=1}^{n} \sum_{\beta=1}^{h} (x_\alpha - x_{\alpha-1})(y_\beta - y_{\beta-1}) \; f(P^*_{\alpha\beta}) =$$

$$= \sum_{\alpha=1}^{n} (x_\alpha - x_{\alpha-1}) \left( \sum_{\beta=1}^{h} (y_\beta - y_{\beta-1}) \; f(P^*_{\alpha\beta}) \right) =$$

$$= \sum_{\alpha=1}^{n} (x_\alpha - x_{\alpha-1}) \underbrace{\left( \sum_{\beta=1}^{h} (y_\beta - y_{\beta-1}) \; f(x_\alpha, y_\beta) \right)}$$

è somma di Riemann della funzione della sola $y$:

$$f(x_\alpha, y)$$

[illegible]

$$\int_c^d f(x_\alpha, y)\,dy$$

$$\rightarrow \quad \sum_{\alpha=1}^{n} (x_\alpha - x_{\alpha-1}) \int_c^d f(x_\alpha, y)\,dy$$

che è somma di Riemann della funzione della sola $x$

$$\int_c^d f(x,y)\,dy$$

che quando $|\Delta_x| \rightarrow 0$

$$\rightarrow \quad \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx$$

Potevo scambiare e trovare $\displaystyle\int_c^d \left( \int_a^b f(x,y)\,dx \right) dy$

Quindi trovo :

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx =$$

$$= \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

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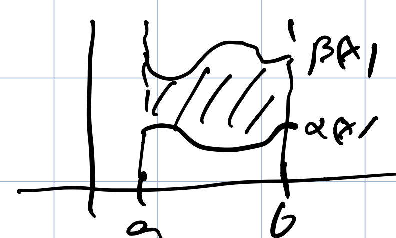
D dominio y-semplice




$$D = \{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ \alpha(x)\le y\le \beta(x)\}$$

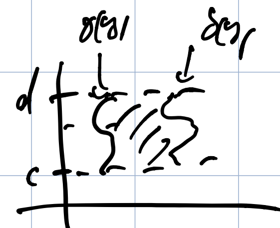
es:


$\alpha(x) = -\sqrt{r^2-x^2}$ $\beta(x) = \sqrt{r^2-x^2}$

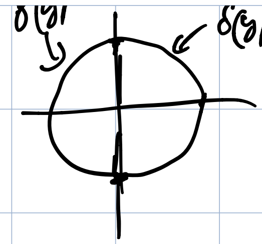
$a = -r$

$b = r$

D x-semplice




$$D = \{(x,y)\in\mathbb{R}^2 \mid c\le y\le d,\ \gamma(y)\le x\le \delta(y)\}$$




$c = -r$

$d = r$

$\gamma(y) = -\sqrt{r^2 - y^2}$

$\delta(y) = \sqrt{r^2 - y^2}$

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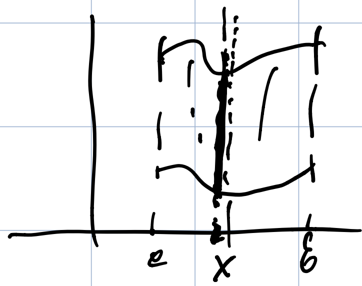


$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left( \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$

caso y-sempl.

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left( \int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx \right) dy$$


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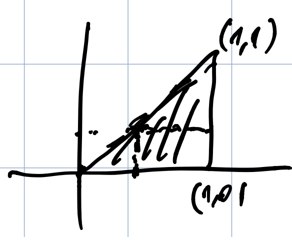
T: 


Calcolare $\iint_T xy\,dx\,dy =$

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$$\int_a^b \left( \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \right) dx = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy$$

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$$= \int_0^1 dx \int_0^x xy\, dy = \int_0^1 dx \; x \left[ \frac{y^2}{2} \right]_0^x =$$

$$\int_0^1 \frac{x^3}{2}\, dx = \left[ \frac{x^4}{8} \right]_0^1 = \frac{1}{8}$$

esercizio: scambiare l'ordine degli integrali

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$$R = \{ 0 \le x \le a,\ 0 \le y \le b \}$$

$$\iint_R x^2 y^2\, dA = \int_0^a dx \int_0^b x^2 y^2\, dy =$$

$$= \int_0^a x^2 \left[\frac{y^3}{3}\right]_0^b dx = \frac{b^3}{3}\left[\frac{x^3}{3}\right]_0^a = \frac{a^3 b^3}{9}$$

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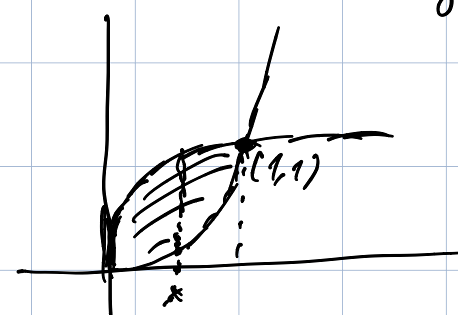
$$\iint_R x y^2\, dA =$$

R $x \geq 0$, $y \geq 0$ $y = x^2$, $x = y^2$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} x y^2\, dy =$$




$$= \int_0^1 x \left[\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 x\left(\sqrt{x}^3 - x^6\right) dx = \cdots$$

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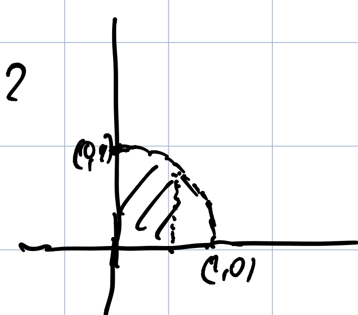
$$\iint_D x \cos y\, dA =$$

$y = 1 - x^2$



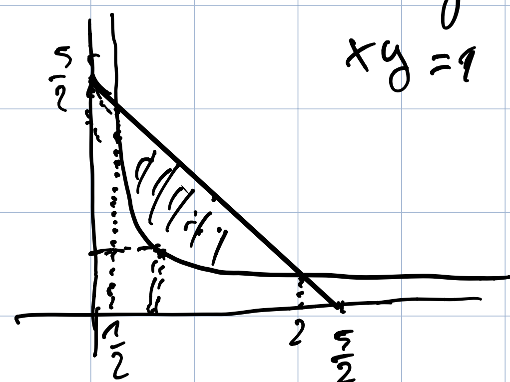

$$= \int_0^1 dx \int_0^{1-x^2} x \cos y\, dy = \int_0^1 x \left[\operatorname{sen} y\right]_0^{1-x^2} dx =$$

$$= \int_0^1 x \, \mathrm{sen}(1-x^2)\, dx = \ldots\ldots$$

$$\iint_D \log(x)\, dA =$$

$2x+2y=5$ 1° quadrante

$xy=1$



calcolo le coordinate delle intersezioni $\begin{cases} 2x+2y=5 \\ xy=1 \end{cases}$

$$2x^2+2-5x=0 \qquad x=\frac{5\pm\sqrt{25-16}}{4} = \begin{cases} 2 \\ \frac{1}{2} \end{cases}$$

$$= \int_{\frac{1}{2}}^{2} dx \int_{\frac{1}{x}}^{\frac{5}{2}-x} (\log x)\, dy = \int_{\frac{1}{2}}^{2} (\log x) \left[ y \right]_{\frac{1}{x}}^{\frac{5}{2}-x} dx =$$

$$= \int_{\frac{1}{2}}^{2} \left(\frac{5}{2} - x - \frac{1}{x}\right) \log|x| \, dx = \dots\dots\dots$$

coordinate polari $\begin{cases} x = \rho \cos\theta \\ y = \rho \,\mathrm{sen}\,\theta \end{cases}$

cambiamento di variabili $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

$$\begin{array}{ccc} \mathbb{R}^2_{u,v} & \longrightarrow & \mathbb{R}^2_{x,y} \\ \cup & & \cup \\ D' & \longrightarrow & D \end{array}$$ invertibile

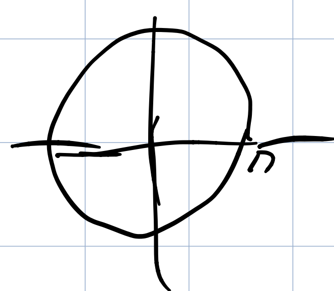
$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$ jacobiana. Sarà invertibile (cioè $\det J \neq 0$)

es: coord. polari $J = \begin{bmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \end{bmatrix}$ $\det J = \rho$

Formule cambiamento di variabili:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v),y(u,v))\,|\det J|\,du\,dv$$

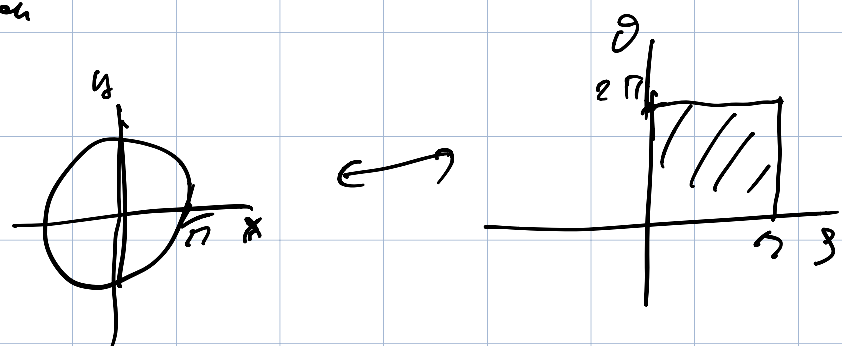
esempio




Area cerchio (raggio r)

$$\text{Area}(C) = \iint_C 1\,dx\,dy = \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 1\,dy = \dots$$

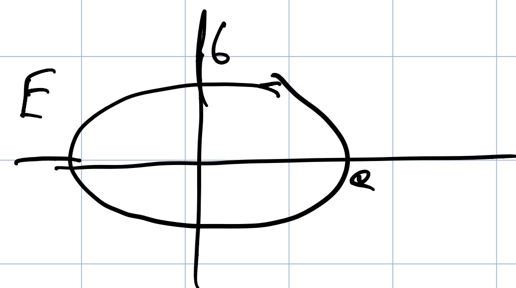
coord polari




$$= \int_0^{2\pi} d\theta \int_0^{r} \rho\,d\rho = \int_0^{2\pi} \left[\frac{\rho^2}{2}\right]_0^r d\theta =$$

$$= \int_0^{2\pi} \frac{r^2}{2}\, d\theta = \frac{r^2}{2}\, 2\pi = \pi r^2$$

Area ellisse




$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$$

$$A = \iint_E 1 \, dx\, dy =$$

$$\frac{x}{a} = u \qquad \frac{y}{b} = v$$

$$= \iint_{E'} ab \, du\, dv =$$

$$E' = \{u^2 + v^2 \le 1\}$$

$$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \det J = ab$$

$$= ab \iint_{E'} 1 \cdot du\, dv = \pi a b$$