IS MUSIC RELEVANT FOR THE HISTORY OF SCIENCE?

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SUMMARY.

I maintain the following thesis: music was the original mathematical model for natural sciences in the West. To support this, arguments and numerous examples could be presented, from the Pythagoreans to Kepler, ..., from Newton to d'Alembert. It is to be noted that the original idea of harmony included the whole of nature and did not leave out human beings, medicine, or life sciences.

THE OLDEST OF ALL QUANTITATIVE PHYSICAL LAWS.

My aim here is simply to illustrate a thesis, which, in its strongest and clearest form, may be stated as follows:

Music was one of the original mathematical models for the sciences of nature in the West.

This thesis is suggested by one of the most ancient events of which man has memory, so ancient that it has entered the realms of myth, and become lost amid the desert wastes. A relationship exists between the length of a string that emits sounds when it is plucked and the pitch of those sounds, as perceived by the ear. This relationship was described by means of a precise mathematical formula, that of inverse proportionality. After Descartes, this was to be written in the form

$$\nu \propto \frac{1}{l}$$

where the pitch, after Mersenne, is measured as the frequency ν and the length is l.

This relationship has remained unchanged until today, for about 2500 years. Is it the only mathematical natural law which is still considered to be valid? Whereas perhaps others may have been modified over the years several times? "... possibly the oldest of all quantitative physical laws" wrote C. Boyer[1]. If so, that "possibly" could be deleted.

The sounds emitted by instruments, that is to say, the musical notes, as heard by the ear, could thus be classified by the proportions 1 : 2, 2 : 3, 3 : 4. These corresponded to the *diapason* interval (the octave, *do - do*), the *diapente* (the fifth, *do - sol*) and the *diatessaron* (the fourth, *do - fa*). The proportion thus established that the greatest importance was to be attributed, not to the single isolated sound, but to the relationship between sounds. At this point the story became even more fascinating, and also relatively well documented, because during the whole of the subsequent evolution of the sciences, controversies continually arose over two main problems. How was the octave to be divided into notes? Which of the relative intervals were to be considered 'consonant', that is to say, 'pleasant', and thus to be included in pieces of music, and which were 'dissonant'? And why? The constant presence of contrasting answers to these questions allows us to consider the sciences as an integral part of our culture, which regularly thrives on the various connections between debates of this kind.

Anyway, in view of the surprising success of our original model of a mathematical law, it was linked with other regular features, identified here and there, and posited as the explanation for other phenomena. The most famous of these was undoubtedly the movement of the planets and the heavenly bodies, thus giving rise to the so-called music of the heavenly spheres. This original seal, this original aporia of its foundation, remained visible for a long time. All, or almost all, the figures that we usually take into consideration in the evolution of the mathematical sciences wrote about these problems. Sometimes they made original contributions. At other times, they simply repeated, with a few personal variations, what they had learnt from tradition. It might be called Pythagorean tradition, from the name of the legendary founder to whom the original discovery was attributed, or Platonic or neo-Platonic, or else it might be placed alongside the rival tradition of Aristoxenos. In any case, many felt they were obliged to render homage to it, in the form of commentaries, compendia, diverse quotations, or actual theories.

Without stopping to think at length, and as a result forgetting others, Euclid[2], Plato[3], Ptolomy[4] and Aristoxenos[5] come to mind. These were followed, skipping the medieval era, before and during the scientific revolution of the seventeenth century, by Maurolico[6], Benedetti, Stevin, Kepler, Galileo Galilei, Mersenne, Huygens, Descartes and Wallis. And the interest in the division of the octave into a certain number of notes, the interest in explaining the consonances, passed intact, or almost intact, through the epoch-making substitution of the Copernican system in the place of the Ptolemaic one. It could be interpreted variously as a theory of music, or as acoustics, as the music of mathematics or the mathematics of music. Under whatever guise, it recurred incessantly in Leibniz, Newton, a member (members?) of the Bernoulli family, D'Alembert and Euler. We shall present only a few examples here, referring the reader to the bibliography for the others.[7]

KEPLER AND LEIBNIZ.

In 1619, Johannes Kepler published his *Harmonices Mundi Libri Quinque*.[8] Starting from the curious idea of continuing Claudius Ptolemy's *Harmonica*, he wrote a complex treatise on geometry, music, astrology and astronomy. His theory of the division of the octave and his justification of consonances is highly original, because it is based on an idea linked with the 'constructibility' of regular polygons with a certain number of sides. In his fifth book, he included his third law of the movement of planets, stating that a sesquialter proportion exists between their periods of rotation T and their

Figure 1.

mean distances from the sun R:

$$T \propto R^{\frac{3}{2}}.$$

This famous result was thus included in a book about the theory of music.

On April 17, 1712, Leibniz wrote a letter to Christian Goldbach.[9] We here present a facsimile of the letter (Fig. 1), together with the translation of parts III, IV e V.

"III. It cannot be excluded that animals may somewhere exist, which have a greater musical sensitivity than us, and take delight at musical proportions which impress us to a lesser extent. I believe, however, that a greater sensitivity of our senses would be more harmful than profitable for us; we would have to hear many unpleasant things, together with our sight, our sense of smell and of touch; and those who have too fine a sensitivity to music are offended by certain deviations which cannot easily be avoided in the practical construction of organs, by which, however, the listeners are not usually offended. We do not count beyond five in music, like those peoples who have not even got beyond three in arithmetic, to whom the German saying about the simpleton is applicable: *er kan nicht über drey zehlen* [he can't count beyond three]. All our usual intervals are ratios based on two of the prime numbers, 1, 2, 3 and 5. If we were endowed with a little more subtlety, we might arrive at the prime number 7. And actually I believe the following ones are also given. Thus the ancients did not openly avoid the number 7. But hardly anybody proceeded as far as the following prime numbers, 11 and 13.

IV. On the contrary, I believe that the basis of consonance is to be sought in the congruence of beats. Music is a recondite arithmetical exercise carried out by the soul, which is unaware that it is counting. It performs many things with a confused, or insensitive perception, which it cannot discern with a clear understanding. Those who think that nothing can exist in the soul, of which the soul itself is not aware, are mistaken. Thus, although the soul does not realise that it is counting, it feels, nevertheless, the effects of this unconscious numbering, or a pleasure at the consonances and a distaste at the dissonances which are the result. The pleasure is aroused by several unconscious congruences. Those who attribute only conscious operations to the soul, in truth, usually make a bad judgement. This has given rise to many errors, not only with philosophers of the past, but also with the followers of Descartes themselves, and with others who are more recent, such as [John] Locke and [Pierre] Bayle. But, to return to our subject, in the octave, every other beat in one series of beats coincides with any beat in the other series. In the fifth, there is a concordance between every third beat in one series and the second of the other one. The polygons of the circle, regular bodies and others of this kind admitted by the highly intelligent Kepler are used $\alpha'\pi\rho o\sigma\delta\iota o'\nu\upsilon\sigma\alpha$ [inopportunely], seeing that the arithmetic of rational numbers does not apply to them.

V. I do not believe that irrational ratios are pleasing to the soul in themselves, except when they are very close to the rational ones which give pleasure. In some cases, however, dissonances are sometimes pleasing, and are used with profit, placed between attractive intervals, just as shadows are contrasted with order and light, so that even more pleasure may be taken in order. Your friend whose note you sent me, who chooses to divide the monochord into the extreme and medium ratios, would actually find intervals which proceed in such a direction as almost to coincide with the major and minor sixth, that is to say, if $AB : BC = BC : CA$, it will be approximately

$$AB - - - - - - - BC - - - - - - - - CA$$

$$8 - - - - - - - - -5 - - - - - - - - -3$$

$$minor \quad sixth - -major \quad sixth.$$

These are as close, approximately, as 809 . 500 . 309, if we admit that $\sqrt{5}$ is 2.236, that is to say, 2.236 to the thousandth and exactly the same as $\sqrt{5}+1$, 2, $\sqrt{5}-1$, whereas $\sqrt{5}$ is wholly irrational and slightly higher than 2.236, but the error is less than $\frac{1}{1000}$. And thus, if that division held in it something pleasing, it would be taken from these close intervals."

EULER AND D'ALEMBERT.

Euler's letter to Johann Bernoulli dated May 25 (June 5) 1731[10] is equally interesting.

"My long delay in writing to Your Excellency and in assuring you of my respectful homage, is due to the fact that for some time now, I have not worked at anything that I thought worthy of being sent to you. For almost all this period, I have dedicated my energies to the construction of a *Systematis Musici*, which I have now almost finished. I therefore take the liberty of sending Your Excellency what I believe I have thus achieved, because I imagine that you may be curious about it, as I recently heard from your esteemed son [Daniel] that you had received news of it from Herr [Jakob] Hermann. My aim, in this work, was to try to develop music as a part of mathematics, and to derive in an orderly fashion, from the right foundations everything that a combination and a mixture of notes may make pleasing. For this complete treatise, I have felt the need of a metaphysical foundation on which to base the reason why one may take delight in a piece of music, and why something is pleasant for one person, but unpleasant for another. I have demonstrated that with notes, the following situation arises: we appreciate many notes combined together if we recognise the ratio of high and low pitch between the notes, in other words *rationem intervallorum pulsuum* [the proportion of the beats of the intervals] given by the strings. And I have also devised the rules for this, to explain how the notes should be put together so that an intelligent ear may take delight in them. But there is yet another reason why we take delight in a piece of music, and this lies in the duration of the notes, and consequently we may even appreciate a piece of music only because we grasp the ratio between the durations of the notes. According to the first *principio*, it is the high or low pitch that we appreciate, but according to the other one, it is their different duration. In order to be perfect, music must combine both and thus we appreciate both *ratione acuminis et gravitatis* and *ratione duratione sonorum* [in view of the ratio between high and low, and that of the duration of the sounds]. The Choral is a kind of music which we appreciate only in the first manner, as it is based only on the harmony of voices, without considering the duration. But an *Exempel* of music which can give pleasure only in the second way is the sound of drums, where only the duration of notes is considered, and not their pitch. Initially I have described music in accordance with the first *principio*, and only subsequently in accordance with the second one, and finally I will deal with both of them together.

In the first part, which is undoubtedly the most *fürnehmste* [important?], the following elements are presented and examined. First, the *accorten* [chords], where I have demonstrated how two or more notes should be used in order to create a pleasant

Harmonie when they are played together. Secondly, I have studied how two chords should be used, so that when they are played one after the other, they sound pleasant. Thirdly, I have considered a whole succession of chords, demonstrating what is needed for them to sound pleasant. The limits within which a succession of this kind may be found constitute the *modos* that I have subsequently described. In this same part, I have also shown what notes are needed on instruments for each *modum*, finding the *genus chromaticum* with 12 notes in an octave very attractive. In the following discussion, I have considered everything in accordance with the *Systemate chromatico*. After this, I have done something more particular than the preceding and I have not applied myself any more to *praxin*. And as I have at the same time prepared a *perfectam enumerationem modorum*, it may be seen that much more progress could be made in music, and that out of all the various *modis*, only two are at present in fashion. These are included in the music that is based on the *Genus chromaticum*, and if it were desirable to use other notes in addition to the 12 found in an octave, or in their place, then it would be possible to obtain infinitely many different types of music.

But all this I have found starting from the first *principio*. Let many different notes be taken, whose *numeri pulsuum* [numbers of beats], that occupy the same period of time, stand to each other as the whole numbers a, b, c, d etc., by which the same notes are usually expressed. Let A be the *minimus communis dividuus* [lowest common multiple]. I call this number the *exponentem* of the same notes, because it is on this basis that we recognise grace that is produced when the same notes are played either at the same time or in succession. I have thus devised a *gradus suavitatis*, the first of which includes the most perfect chord, that is to say, when all the notes are relatively equal. The following ones include the less perfect, according to their order. From the *exponente* it is possible to recognise the *gradus suavitatis* in the following manner: I break it down into its *factores simplicissimos* [most simple factors], I add these together, and then I subtract from their sum $n-1$ (n stands for the *numerum factorum* [number of factors]); the number that is thus obtained represents the *gradum*. For example, let the notes be 1, 2, 3, 4, 6, then the *exponens* of these notes will be 12. The *factores simplicissimi* of this are the following three: 2.2.3, the sum of which, less 2, gives 5. It is clear from this that the harmony of these notes will be pleasant to the 5^{th} degree. In this way, it is possible to find the *exponentem* of a whole piece of music, if all the notes are expressed as whole numbers, and the *minimum commune dividuum* is taken. But this *exponens* is nothing other than the *modus musicus* of the piece itself. As all the notes are present on the clavichord, the *exponens* of a piece is $2^n.3^3.5^2$, where n depends on the number of octaves. Consequently, the following 12 notes must be present in an octave:

$$480, 512, 540, 576, 600, 620, 675, 720, 768, 800, 864, 900$$

the thirteenth note is one octave higher than the first, that is to say, 960.

This is what I wanted to let Your Excellency know about my *Musica Theoretica*. ....".

Johann Bernoulli's reply was dated August 11 (July 31).[10]

"It was a great pleasure for me to learn that you are working on the completion of a *Systematis Musici* (which must be almost ready); I have no doubt that it will be a fine work when it is published, and will offer a sufficient demonstration of the excellent *ingenium* of its *Autoris*; I can well imagine that nobody else will attempt a similar *opus*, in which everything is derived from a mathematical basis, seeing that there are few, if any, *Scriptores Musici* endowed with such a great and unlimited mathematical understanding as the *Herr Professor*. I therefore look forward to seeing your work. I certainly could not imagine what that basis may be which must be so metaphysical, as you say, and through which it may be possible to explain why a piece of music gives pleasure, and why something sounds pleasant to us, and unpleasant to another: of course, we have a general idea of harmony, that it is pleasant when it is well founded and the consonances are well organised; then, I note that also dissonances are useful in music, with the result that the beauty of the consonances which follow them is improved, in accordance with the common proverb, *opposita juxta se posita magis elucescunt* [opposites which are placed next to each other stand out more clearly], and this is also connected with the use of shadows in painting, which serves to let the light stand out. In *musica practica*, I believe it depends above all on the type and modification [modulation?] to which one is accustomed and this type depends largely on the nature and temperament of people; as a result, some think that this piece is sweet and pleasant, and others that other piece, and consequently, the type of Italian music does not agree with French music, and the latter does not agree with English music, etc. In a word, *de gustibus non est disputandum.* Thus, if one desires to base the pleasantness of a piece of music on its nature itself, it is necessary to define what one means by pleasantness and not say that this or the other piece is pleasant because I like it, another might not like the same piece; for example, Midas preferred the musical din of Pan to the sound of the Apollo's lyre. The *Herr Professor* say that it is possible to judge the pleasantness or unpleasantness of 4 notes played together, by examining the ratio between the high or low pitch of the notes, that is to say, the *rationem intervallorum pulsuum* [the ratio between the beats of the intervals] given by the strings. From this, you have derived the rule which establishes how the notes are to be combined, so that an intelligent ear can take delight in them. I think that this is appropriate for a musician who is more concerned about the accuracy of a piece of music than its effect, which satisfies the listener; a person of this kind will undoubtedly find enjoyment and delight, if you have written this down and examine it and find that it is well composed in accordance with the fundamental rules; but as a piece of music is usually played to ears that are devoid of understanding, and are not able to recognise the *rationem intervallorum pulsuum* of the strings, and are even less able to count, then I believe that the same ears will appreciate or refuse the same piece of music, depending on whether they are used to this or that kind of music. I greatly appreciate your project because at least in this way the *Theoria Musices* is improved, and it is demonstrated that a *Mathematicus* is able to deal with almost all the sciences, whereas, on the contrary, the other masters who are only *practici* are as capable of writing of their art as a blind man is of colours.

When this *Tractatus Musices* is finished, you will undoubtedly give serious attention to the *Mechanicam* that you have planned (of which I have been informed in a letter); I expect something special from this ...".

The *Tractatus Musices* was to be entitled *Tentamen novae theoriae musicae* and was finally published only in 1739,[11] after the *Mechanicam* of 1736.

In 1752, there was a brief exchange of letters between Euler and Jean Philippe Rameau, the famous musician and theoretician.[11] But it was rather Jean D'Alembert who was the main interlocutor of this French composer, who based (tonal) harmony on the harmonics of sound. In actual fact, the mathematician and encyclopaedist dedicated a whole book to the problem *Élémens de musique Théorique et pratique suivant les principes de M. Rameau*,[12] which was published in the same year, 1752.

The part that D'Alembert played in the famous *querelle des bouffons* in connection with Italian music is fairly well known.[13] I believe, however, that it will come as a surprise to many to learn what decisive argument he used to back up his thesis. In about 1752, he wrote: "I speak from experience. I have examined many recitative scenes; I have sung them in the Italian style, leaving out the long cadences and the held notes: the result was a song very similar to Italian recitative, and I would dare say that it would not be unpleasant, however little inclined one may be. I carried out this experiment in the presence of some other people, who reached the same conclusion as myself".[14]

Under the entry "Fondamental"[15] in the *Encyclopèdie*, he dealt with the basic sound and the first harmonics which form the perfect major chord. If the basic sound is a *do*, these will include the repetition of *do* in the higher octave, *mi* and *sol*. Consequently, he also criticised the solutions considered by Daniel Bernoulli and Euler for the equation of a vibrating string. In view of its importance in the evolution towards the notion of a function, as the solution of a differential equation, this controversy is one of the most widely studied by historians of mathematical sciences. However, in the best known current reconstructions, there is no longer any trace of music. And yet it is not necessary to read the 'frivolous' (frivolous?) entry in the *Encyclopèdie* to realise this.

The wave equation proposed by D'Alembert was to become the main explanatory model in mathematical physics. The dynamic phenomenon to be studied was translated into a differential equation (from now on also with partial derivatives); the solutions were calculated as functions of space and time, and from them the properties of the phenomenon could be read. This method of explaining and calculating was to be the main one used, at least up to Poincaré. Only in 1912, faced with new quantum phenomena, which appeared to impose the well-known discontinuities, did the latter comment disconsolately: "physical phenomena would cease to obey laws that can be expressed by means of differential equations, and this would undoubtedly be the biggest, and most wide-reaching revolution in natural philosophy since the time of Newton".[16]

D'Alembert opened the period dominated by differential equations in the modern sense in 1747, with his famous article, "Recherches sur la courbe que forme une corde tenduë mise en vibration".[17] This was followed in 1750 by an "Addition", at the end of which his polemical stance against the rival musical theory was quite clear.

"It is clear from the preceding formulae that, given an equal tension and thickness, the number of vibrations in the same time is inversely proportional to the length of the strings. As the higher or lower sound of the strings depends on their larger or smaller number of vibrations in the given time, it is undoubtedly for this reason that some very capable modern authors have considered it possible to represent the sounds by means of the logarithms of the ratios between the lengths of the strings. This idea is ingenious, and would appear to be based equally on figures of speech in acoustics and music, when we say that if four strings a, b, c, d are geometrically proportional, the interval formed by sounds a and b will be *equal* to the interval formed by c and d; hence it was considered possible to conclude that the logarithms of the relationships $\frac{a}{b}$ and $\frac{c}{d}$ represented the intervals between the sounds. But undoubtedly this conclusion was not claimed to be anyhting more than a purely arbitrary supposition; the words *interval* between sounds, *equality* and *difference of intervals* are only abbreviated figures of speech, which should not be given a wider meaning than they really have. Sounds are merely sensations, and consequently they do not in reality have any ratio with one another; sounds cannot be compared, any more than colours can; all that is needed is a little attention to hear this ...".

Thus the French mathematician and physicist believed that this controversy in the field of music was pertinent to mathematical studies on vibrating strings. (The *corpus callosum* of his brain had not been cut, and the right hemisphere, that of music, communicated perfectly with the left hemisphere, that of language.) It was not difficult for him to believe this, because the harmonic sounds, placed by him and by Rameau at the basis of music, appeared to undermine the millenary Pythagorean tradition based, instead, on numerical proportions. Euler, on the other hand, appeared to prefer to stay within this tradition, even if he wanted to extend its rigourousness and its application, making musicians count up to 7! The controversy about vibrating strings thus saw the participation of people who were fighting at the same time on behalf of the relative musical theories.

SINGULAR COINCIDENCES.

We might be surprised at the singular coincidence between a scientific controversy and a musical one, which arose in the century of the Enlightenment. And yet this coincidence was not new, and was not the only one.

Let us come back to the original seal of our story, and let us carefully consider an important detail. Thanks to the proportions chosen for the octave, the fifth and the fourth, the Pythagorean musical school was rapidly able to calculate the tone $fa - sol$ as the difference between the fifth $do - sol$ and the fourth $do - fa$, and consequently as the ratio $\frac{3}{2} : \frac{4}{3} = \frac{9}{8}$. All the treatises on music at that time were taken up with the question whether it was possible to divide the tone into two equal parts (semitones). The Pythagorean tradition denied that it was possible, but the followers of Aristoxenos readily admitted that it was. Why? Dividing the Pythagorean tone into two parts would mean admitting the existence of the proportional mean between 9 and 8, that is to say, $9 : \alpha = \alpha : 8$, where $9 : \alpha$ and $\alpha : 8$ are the proportions of the required semitone. But how much was α to be? Clearly, $\alpha = \sqrt{9.8}$ and therefore $\alpha = 3.2.\sqrt{2}$! Thus the most famous controversy in ancient Greek mathematics, the one concerning irrational numbers, immediately acquired a musical tone. The discussion of the relative model of the numerical or geometrical continuum,[18] variously represented in ancient philosophy by means of the paradoxes of Zeno of Elea, could be directly transferred to music. Anybody who, like Aristoxenos, preferred to look for answers in the practical activity of the earthly world, and on musical instruments instead of in the heaven of Platonic ideas, could not doubt that it was possible to place his finger on the string at exactly the point which corresponded to the division into equal semitones.

As we know perfectly well, the story was to continue to evolve constantly, without ever succeeding in reaching a complete solution, at least not for everybody. It returned, for example, in the margins of a manuscript left by Francesco Maurolico, bearing dates between 1567 and 1570, in the form "$9.r72.8$" where r stood for root. And it was denied that consonant intervals suitable for music could be obtained by means of irrational proportions. Because "Only God is infinite".[6]

At the beginning of the twentieth century, Georg Cantor would have liked to grasp that continuum, but it always slipped through his fingers, with the result that he ended up by losing his mind. He, too, was forced to use the string of a violin to evoke it.[19]

I have found, very recently, only one book on the history of mathematics, published in 1997 by R. Cooke, in which the exercise is assigned of dividing the octave into two equal parts and discussing what the Pythagoreans thought of this idea.[20]

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