LOGICLESS NONSTANDARD ANALYSIS: AN AXIOM SYSTEM

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We give an axiomatic framework for getting full elementary extensions such as ultrapowers. From five axioms, all properties of a nonstandard extension are derived in a rather algebraic manner, without the use of any logical notions such as formulas or satisfaction. For example, when applied to the real number system, it provides a complete framework for working with hyperreals. This has possible pedagogical and expository applications as presented in, e.g., [2, 3], but we avoid use of special logical axioms such as the transfer axiom of [2, 3].

Terminology. An n-ary *partial function f on a set X* is a function whose domain is a subset of X^n and whose range is a subset of X (here $n \in \omega$).

For $n > 0$ and $1 \leq k \leq n$, let $P_k^{X,n}$ be the *k-th n-ary projection function on X*, i.e. the total function $P_k^{X,n} \colon X^n \to X$ satisfying $P_k^{X,n}(x_1, \ldots, x_n) = x_k$.

For $a \in X$, $C_a^{X,n} \colon X^n \to X$ is the n-ary *constant function* taking the value a.

We let f, g, h, etc, denote partial functions.

The Axioms. Let $A \subset B$ be non-empty sets and suppose that for each partial function f on A there is associated a partial function ${}^*f$ on B with the same arity. We refer to ${}^*f$ as the transform of f. The five axioms are:

(1) *The transform preserves projection functions:* ${}^*P_k^{A,n} = P_k^{B,n}$.
(2) *The transform preserves constant functions:* For any $a \in A$, ${}^*C_a^{A,n} = C_a^{B,n}$.
(3) *The transform preserves compositions:* ${}^*(f \circ g) = {}^*f \circ {}^*g$, where f and g are partial functions on A. (Similarly for more general forms of composition.)
(4) *If* $\mathrm{dom}(f)$ *is itself a partial function, say* $\mathrm{dom}(f) = g$*, then* $\mathrm{dom}({}^*f) = {}^*g$.
(5) *If* $\mathrm{dom}(f)$ *is finite then* ${}^*f = f$.

Suppose these axioms are satisfied and fix an element $a \in A$. For each relation R on A, identify R with the partial constant function f_R having domain R and taking the constant value a, and let ${}^*R$ be defined as the domain of ${}^*f_R$.

MetaTheorem. Let L_A be the language which consists of all relations and functions on A, and let $\mathfrak{A}$ be the structure over A where each symbol of L_A is interpreted as itself, and $\mathfrak{B}$ the structure over B where each symbol of L_A is interpreted as its transform. Under the axioms, $\mathfrak{A} \preccurlyeq \mathfrak{B}$, i.e. $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$.

References

[1] Goldblatt, R. *Lectures on the Hyperreals,* Springer, 1998.
[2] Keisler, H. J. *Elementary Calculus: An Approach Using Infinitesimals,* Online Edition, 2000.
[3] Keisler, H. J. *Foundations of Infinitesimal Calculus,* Online Edition, 2007.

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