A REMARK ON ULTRAPOWER CARDINALITY AND THE CONTINUUM PROBLEM

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[Joint work with Aleksandar Jovanović.]

In this work we discuss the relationship between the ultrapower cardinality jumps, the two cardinal properties and the continuum problem (CP). In ZFC the equation $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ we prefer written as $2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}$, naming $f$ the CP displacement (function). We say that $f$ is bounded at $\alpha$ if $f(\alpha) < 2^{\aleph_\alpha}$, and unbounded when $f(\alpha) = F(\alpha) = 2^{\aleph_\alpha}$. There are all those exciting well known results on CP. Some solutions express some preferences towards smaller $f$ (e.g. the case of RV-large cardinals). For an ultrafilter $D$ over $\kappa$, define its cardinality trace as

$$\text{ct}(D) = \{|\prod_D \lambda| : \lambda < \kappa\}$$

and call $D$ jumping when $|\text{ct}(D)| > 1$. For example, when $D$ is regular, it is not jumping, when $\kappa$ is measurable with $D$, $\kappa$-complete, $|\text{ct}(D)| = 2^\kappa$. Magidor constructed models with nonregular jumping ultrafilters over small cardinals, which are hardest to obtain, using large cardinals.

A theory $T$ with unary predicate $U$ admits pair $(\kappa, \lambda)$ if it has a model of cardinality $\kappa$ in which $|U| = \lambda$. A pair $(\kappa, \lambda)$ is a left large gap (LLG) for $T$ if $T$ admits $(\kappa, \lambda)$ but does not admit the pair $(\kappa^+, \lambda)$. Now we can state the theorem relating the mentioned notions.

**Theorem 1.** Let $f$ be the displacement function in the continuum problem, $2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}$. Let $T$ be a theory with $(\aleph_\xi(\lambda), \lambda)$ as LLG for all $\lambda$. Let $\aleph_\sigma^{<\aleph_\alpha} = \aleph_\sigma$ and let $(\aleph_\sigma, \kappa)$ be a LLG for $T$. Let $D$ be a uniform nonregular ultrafilter over $\aleph_\sigma$ with jumps after $\kappa$:

$$\aleph_\eta = |\prod_D \kappa| < |\prod_D \aleph_\sigma|.$$  

Then, $\eta < \sigma + f(\sigma) \leq \eta + \xi \leq \eta + \sigma$, binding CP-jump with the ultrapower cardinality jump and the diameter of the gap.

As examples, we mention some consequences.

1. Let $D$ be a jumping ultrafilter over $\aleph_{17}$ and $\aleph_{17}^{<\aleph_{17}} = \aleph_{17}$. If $|\prod_D \omega| \leq \aleph_{17}$, then $2^{\aleph_{17}} \leq \aleph_{34}$.

2. If $2^{\aleph_{17}} = \aleph_{\omega+1}$, then there is no jumping ultrafilter over $\aleph_{17}$.

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