A REMARK ON ULTRAPOWER CARDINALITY AND THE CONTINUUM PROBLEM

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[Joint work with Aleksandar Jovanović.]

In this work we discuss the relationship between the ultrapower cardinality jumps, the two cardinal properties and the continuum problem (CP). In ZFC the equation $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ we prefer written as $2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}$, naming f the CP displacement (function). We say that f is bounded at α if $f(\alpha) < 2^{\aleph_\alpha}$, and unbounded when $f(\alpha) = F(\alpha) = 2^{\aleph_\alpha}$. There are all those exciting well known results on CP. Some solutions express some preferences towards smaller f ($f = 1$ iff GCH), avoiding wilder possibilities when f is large or unbounded somewhere (e.g. the case of RV-large cardinals). For an ultrafilter D over κ, define its cardinality trace as

$$\mathrm{ct}\,(D) = \{|\prod_D \lambda| : \lambda < \kappa\}$$

and call D jumping when $|\mathrm{ct}\,(D)| > 1$. For example, when D is regular, it is not jumping, when κ is measurable with D κ-complete, $|\mathrm{ct}\,(D)| = 2^\kappa$. Magidor constructed models with nonregular jumping ultrafilters over small cardinals, which are hardest to obtain, using large cardinals.

A theory T with unary predicate U admits pair (κ, λ) if it has a model of cardinality κ in which $|U| = \lambda$. A pair (κ, λ) is a left large gap (LLG) for T if T admits (κ, λ) but does not admit the pair (κ^+, λ). Now we can state the theorem relating the mentioned notions.

Theorem 1. Let f be the displacement function in the continuum problem, $2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}$. Let T be a theory with $(\aleph_\xi(\lambda), \lambda)$ as LLG for all λ. Let $\aleph_\sigma^{<\aleph_\sigma} = \aleph_\sigma$ and let $(\aleph_\sigma, \kappa)$ be a LLG for T. Let D be a uniform nonregular ultrafilter over $\aleph_\sigma$ with jumps after κ:

$$\aleph_\eta = |\prod_D \kappa| < |\prod_D \aleph_\sigma|.$$

Then, $\eta < \sigma + f(\sigma) \leqslant \eta + \xi \leqslant \eta + \sigma$, binding CP-jump with the ultrapower cardinality jump and the diameter of the gap.

As examples, we mention some consequences.

(1) Let D be a jumping ultrafilter over $\aleph_{17}$ and $\aleph_{17}^{<\aleph_{17}} = \aleph_{17}$. If $|\prod_D \omega| \leqslant \aleph_{17}$, then $2^{\aleph_{17}} \leqslant \aleph_{34}$.
(2) If $2^{\aleph_{17}} = \aleph_{\omega+1}$, then there is no jumping ultrafilter over $\aleph_{17}$.

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