COMFORT ORDER ON LOCALLY COMPACT TOPOLOGICAL SPACES

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The Comfort order of the Stone-Čech reminder $\beta S - S$ of discrete space S has often been studied, (See [2]). In this paper we define comfort order on $S^{\mathcal{F}}$(which is the set of all multiplicative mean on $\mathcal{F}$). Throughout this paper S is a locally compact topological Hausdorff space and $\mathcal{CB}(S)$ is the $\mathcal{C}^*$- algebra of all bounded continuous complex-valued functions on S with supremum norm. Let $\mathcal{F}$ be a conjugate closed subalgebra $\mathcal{F}$ of $\mathcal{CB}(S)$ containing the constant functions, then the set of all multiplicative means of $\mathcal{F}$, which denote by $S^{\mathcal{F}}$, equipped with the Gelfand topology is a compact Hausdorff space, furthermore the evaluation mapping $\varepsilon : S \to S^{\mathcal{F}}$, where $\varepsilon(s)(f) = f(s)$ is a continuous map onto a dense subset of $S^{\mathcal{F}}$. Also $\varepsilon^* : \mathcal{C}(S^{\mathcal{F}}) \to \mathcal{F}$ is an isometric isomorphism and $\widehat{f} = (\varepsilon^*)^{-1}(f) \in \mathcal{C}(S^{\mathcal{F}})$ for $f \in \mathcal{F}$ is given by $\widehat{f}(\mu) = \mu(f)$ for all $\mu \in S^{\mathcal{F}}$, (see [1]). For $\mathcal{F}$ (of $\mathcal{CB}(S)$) on a Hausdorff space S, similar to [3], that the compactification $S^{\mathcal{F}}$ can be reconstructed by a suitable quotient space of the set of all z-ultrafilters.

Let X be a topological Hausdorff space and $\widetilde{p} \in S^{\mathcal{F}}$. Let $s \mapsto x_s : S \to X$ be a continuous function and $y \in X$. We say $\widetilde{p} - \mathrm{Lim}_{s\in S}\, x_s = y$ if and only if for every open set U of y, $\widetilde{p} \in (\overline{x^{-1}(U)})^{\circ}$. If $\widetilde{p} - \mathrm{Lim}_{s\in S}\, x_s$ exists, then it is unique and also if X is a compact space then $\widetilde{p} - \mathrm{Lim}_{s\in S}\, x_s$ exists. Let X be a topological Hausdorff space and $\widetilde{p} \in S^{\mathcal{F}}$. We say X is a $\mathcal{F}_{\widetilde{p}}$ (or $\widetilde{p}$) -compact space if $\widetilde{p} - \mathrm{Lim}_{s\in S}\, f(s)$ exist for each continuous function $f : S \to X$. So we can define topological comfort order ($\leq_c$) on $S^{\mathcal{F}}$. For $\widetilde{p}, \widetilde{q} \in S^{\mathcal{F}}$, we define $\widetilde{p} \leq_c \widetilde{q}$ if and only if each $\widetilde{q}$ -compact space be a $\widetilde{p}$ -compact space.

Now let X be a completely regular Hausdorff space, let S be a locally compact topological Hausdorff space. Let $\mathcal{F}$ and $\mathcal{U}$ be conjugate closed subalgebra of $\mathcal{CB}(S)$ and $\mathcal{CB}(X)$, respectively, containing the constant functions. We define

$$(\mathcal{F}, \mathcal{U}, \widetilde{p}) = \bigcap \{Y : X \subseteq Y \subseteq X^{\mathcal{U}},\ \text{Y is a } \widetilde{p} \text{ -compact space.}\}.$$

Lemma 1. *Let $\alpha = |S|$, $\widetilde{p} \in S^{\mathcal{F}}$ and $A_\circ(\widetilde{p}, X) = X$. Inductively, let $\sigma < \alpha^+$ be given. If σ be a non zero limit ordinal, let $A_\sigma(\widetilde{p}, X) = \bigcup_{r<\sigma} A_r(\widetilde{p}, X)$. If $\sigma = \tau + 1$, let $A_\sigma(\widetilde{p}, X) = \{\widetilde{p} - \lim_{s\in S} f(s) : f : S \to A_\tau(\widetilde{p}, X) \subseteq X^{\mathcal{U}}$ be continuous.$\}$. Then $(\mathcal{F}, \mathcal{U}, \widetilde{p}) = \bigcup_{\sigma<\alpha^+} A_r(\widetilde{p}, X)$.*

Theorem 2. *Let S be a locally compact Hausdorff topological space. Let $\mathcal{F}$ be a conjugate closed subalgebra of $\mathcal{CB}(S)$ contained constant functions. Let $\varepsilon : S \to S^{\mathcal{F}}$ be a evaluation map and $|\varepsilon(S)| = \alpha \leq |S|$. Then $|(\mathcal{F}, \widetilde{p})(S)| \leq 2^\alpha$, where $(\mathcal{F}, \widetilde{p})(S) = \bigcap\{Y : S \subseteq Y \subseteq S^{\mathcal{F}},\ Y$ is $\widetilde{p}$ -compact space.$\}$.*

Theorem 3. *Let S be an infinite locally compact Hausdorff topological space. Let $\mathcal{F}$ be a conjugate closed subalgebra of $\mathcal{CB}(S)$ contained constant functions. Let $\widetilde{p}, \widetilde{q} \in S^{\mathcal{F}} - \varepsilon(S)$. Let X is $\widetilde{p}$-compact and every continuous function $f : S \to X$ can be extended to a continuous function $\widehat{f} : S^{\mathcal{F}} \to \beta X$. Then $\widetilde{p} \leq_c \widetilde{q}$ iff $(\mathcal{F}, \widetilde{p})(S) \subseteq (\mathcal{F}, \widetilde{q})(S)$ iff $\widetilde{p} \in (\mathcal{F}, \widetilde{q})(S)$ iff there exists a continuous function $f : S \to (\mathcal{F}, \widetilde{q})(S)$ such that $\widehat{f}(\widetilde{q}) = \widetilde{p} \notin f(S)$ iff $(\mathcal{F}, \widetilde{q})(S)$ is $\widetilde{p}$-compact.*

References

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