Logicless Non-Standard Analysis: An Axiom System

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Getting reals from rationals: Construction vs Axiomatic setup

Reals from rationals: Construction

- Dedekind's method of cuts (order-completion), or
- Cantor's method of using equivalence classes of Cauchy sequences of rationals (metric completion)
- Provides existence proof, and classic techniques

But once the construction is done, no use is ever made of how the reals are constructed! And all we need in practice are the axioms for a complete ordered field:

Reals from rationals: Axiomatic setup

- Axioms for complete ordered fields
- Provides rigorous framework for real numbers
- Avoids getting bogged down with the construction of reals
- Primary approach in many modern real analysis textbooks

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- Useful in developing infinitesimals rigorously without logic, as in some modern calculus texts (Keisler, Crowell)

How to construct proper elementray extensions

- Logical methods (Lowenheim-Skolem / compactness arguments): Not appropriate for non-logicians
- The ultrapower construction (over non-principal ultrafilters):
  1. Avoids logic
  2. Sufficiently algebraic (?) for non-logicians (cf. quotient field from a commutative ring over a maximal ideal)

Getting hyperreals from reals: Construction

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Total and Partial functions, Projections, Composition

- f is an n-ary **total function** on $A \leftrightarrow f : A^n \to A$
- f is an n-ary **partial function** on $A \leftrightarrow f : D \to A,\ D \subseteq A^n$
- f is the **k-th n-ary projection** over A $(1 \le k \le n) \leftrightarrow$ $f : A^n \to A$ and $f(x_1, \ldots, x_n) = x_k$
- **General compositions** (substitutions) of partial functions: Example: If $\phi(x, y, z, w) \equiv f(x, g(y, z), h(w))$, then ϕ is a composition of f, g, h

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Extending the collection of all partial functions on a set

- A : A fixed set, together with the collection of all partial functions on A
- B : A proper superset of A, i.e. $A \subsetneq B$
- The transform: To every partial function f on A, there is associated a partial function ${}^*f$ on B with the same arity, called the transform of f

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The Five Axioms

- **Axiom 1 (Projection Function Axiom).** If f is a projection over A, then ${}^*\!f$ is the corresponding projection over B
- **Axiom 2 (Constant Function Axiom).** If f is a constant function over A, then ${}^*\!f$ is the constant function over B with the same arity and taking same constant value as f
- **Axiom 3 (Composition Axiom).** Composition of partial functions are preserved ${}^*(f \circ g) = {}^*\!f \circ {}^*g$, where f and g are partial functions on A; and similarly for more general forms of composition
- **Axiom 4 (The Domain Axiom).** If the domain of a partial $(n+1)$-ary function f is itself a partial (n-ary) function g, then $\mathrm{dom}({}^*\!f) = {}^*g$
- **Axiom 5 (The Finiteness Axiom).** Finite functions are invariant: If $\mathrm{dom}(f)$ is finite then ${}^*\!f = f$

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Defining transforms of relations

- Fix $a \in A$
- Given a relation R on A (i.e. $R \subseteq A^n$), identify R with the partial constant function f_R having domain R and taking the constant value a
- Let ${}^*R$ be defined as the domain of ${}^*f_R$
- This definition of ${}^*R$ is independent of the choice of the element $a \in A$, assuming that axioms 1–5 hold

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An axiomatic approach to full elementary extensions

Let

1. $L_A =$ The language which consists of all relations and functions on A
2. $\mathfrak{A} =$ The structure over A where each symbol of L_A is interpreted as itself
3. $\mathfrak{B} =$ The structure over B where each symbol of L_A is interpreted as its transform

Then, under axioms 1–5, we have: $\mathfrak{A} \preccurlyeq \mathfrak{B}$, i.e. $\mathfrak{A}$ must be an elementary substructure of $\mathfrak{B}$.