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RELATIVE SET THEORY

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Axioms of FRIST

Primitive concepts: $\in$, $\sqsubseteq$

We postulate all axioms of ZFC
(with Separation and Replacement for $\in$-formulas).

$\mathcal{P}^{\mathbf{fin}}A$ is the set of all finite subsets of A.

We read $x \sqsubseteq y$ as "x appears at the level of y".

Notation:
$x \in \mathbf{V}(y)$ means $x \sqsubseteq y$.

$x \in \mathbf{V}(x_1, \ldots, x_k)$ means $x \in \mathbf{V}(x_1) \vee \cdots \vee x \in \mathbf{V}(x_k)$.

$\mathbf{V}(x) \subseteq \mathbf{V}(y)$ means $(\forall z)(z \in \mathbf{V}(x) \Rightarrow z \in \mathbf{V}(y))$.

$\mathbf{V}(x) \subset \mathbf{V}(y)$ means $\mathbf{V}(x) \subseteq \mathbf{V}(y) \wedge \neg \mathbf{V}(y) \subseteq \mathbf{V}(x)$.

Relativization:

$\sqsubseteq$ is a dense total pre-ordering with a least element 0 and no greatest element.

In detail, the conjunction of the universal closures of:

$0 \in \mathsf{V}(x) \wedge x \in \mathsf{V}(x)$;

$y \in \mathsf{V}(x) \Rightarrow \mathsf{V}(y) \subseteq \mathsf{V}(x)$;

$\mathsf{V}(x) \subseteq \mathsf{V}(y) \ \vee \ \mathsf{V}(y) \subseteq \mathsf{V}(x)$;

$(\exists y)\ \mathsf{V}(x) \subset \mathsf{V}(y)$;

$\mathsf{V}(x) \subset \mathsf{V}(y) \Rightarrow (\exists z)(\mathsf{V}(x) \subset \mathsf{V}(z) \subset \mathsf{V}(y))$.

Let $\mathcal{P}(x_1, \ldots, x_k)$ be an $\in$-$\sqsubseteq$-formula and α a variable (the possibility that it is one of the variables $x_1, \ldots, x_k$ is allowed).

$\mathcal{P}^\alpha(x_1, \ldots, x_k)$ is the formula obtained by replacing each occurrence of $\mathbf{V}(\cdot)$ in $\mathcal{P}(x_1, \ldots, x_k)$ by $\mathbf{V}(\cdot, \alpha)$.

In terms of $\sqsubseteq$ this means that every occurence of $\sqsubseteq$ is replaced by $\sqsubseteq_\alpha$, where $x \sqsubseteq_\alpha y$ means $x \sqsubseteq \alpha \vee x \sqsubseteq y$.

Explicitly:

- $(x \in y)^\alpha$ is $(x \in y)$;
- $(x \in \mathbf{V}(y))^\alpha$ is $(x \in \mathbf{V}(y, \alpha))$;
- $(x = y)^\alpha$ is $(x = y)$;
- $(\mathcal{P} \wedge \mathcal{Q})^\alpha$ is $(\mathcal{P}^\alpha \wedge \mathcal{Q}^\alpha)$, and similarly for the other connectives;
- $(\forall x \mathcal{P})^\alpha$ is $\forall x\ (\mathcal{P}^\alpha)$ and $(\exists x \mathcal{P})^\alpha$ is $\exists x\ (\mathcal{P}^\alpha)$.

Intuitively, $\mathcal{P}^\alpha$ makes the same statement about the level $\mathbf{V}(\alpha)$ as $\mathcal{P}$ makes about $\mathbf{V}(0)$.

Definition. A formula $\mathcal{Q}(x_1, \ldots, x_n)$ is **internal** if it is of the form $\mathcal{P}^{x_1, \ldots, x_k}(x_1, \ldots, x_k)$, for some $\in$-$\sqsubseteq$-formula $\mathcal{P}(x_1, \ldots, x_k)$.

Transfer:

If $\mathbf{V}(\alpha) \subseteq \mathbf{V}(\beta)$ and $x_1, \ldots, x_k \in \mathbf{V}(\alpha)$, then

$$\mathcal{P}^{\alpha}(x_1, \ldots, x_k) \iff \mathcal{P}^{\beta}(x_1, \ldots, x_k).$$

Informally:
A statement with parameters from some level is true about this level if and only if it is true about any finer level.

Corollaries:

Existential Closure Principle

Given a formula $\mathcal{P}(x, x_1, \ldots, x_k)$ in the $\in$-language:
If $(\exists x)\ \mathcal{P}(x, x_1, \ldots, x_k)$ is true, then
$(\exists x \in \mathbf{V}(x_1, \ldots, x_k))\mathcal{P}(x, x_1, \ldots, x_k)$ is true.

Universal Closure Principle

Given a formula $\mathcal{P}(x, x_1, \ldots, x_k)$ in the $\in$-language:
If $(\forall x \in \mathbf{V}(x_1, \ldots, x_k))\mathcal{P}(x, x_1, \ldots, x_k)$ is true, then
$(\forall x)\ \mathcal{P}(x, x_1, \ldots, x_k)$ is true.

Standardization:

Given any α and any $x_1, \ldots, x_k$;
For every A there is $B \in \mathbf{V}(\alpha)$ such that for all $z \in \mathbf{V}(\alpha)$

$$z \in B \iff z \in A \wedge \mathcal{P}^{\alpha}(z, A, x_1, \ldots, x_k).$$

Corollaries:

For every A there is $B \in \mathbf{V}(\alpha)$ such that for all $z \in \mathbf{V}(\alpha)$ $z \in B \iff z \in A \quad$ (α-*shadow of* A).

Neighbor Principle

If a real number is not superlarge relative to a given level, then there is a real number appearing at that level and superclose to it (relative to that level).

Definition Principle

1. Let $\mathcal{P}(x, A, x_1, \ldots, x_k)$ be an internal formula.
   Then there exists a set $B \in \mathbf{V}(A, x_1, \ldots, x_k)$ such that
   $(\forall x)\,(x \in B \iff x \in A \wedge \mathcal{P}(x, A, x_1, \ldots, x_k))$.

2. Let $\mathcal{P}(x, y, A, x_1, \ldots, x_k)$ be an internal formula.
   If
   $$(\forall x \in A)(\exists! y)\mathcal{P}(x, y, A, x_1, \ldots, x_k),$$
   then there is a function $F \in \mathbf{V}(A, x_1, \ldots, x_k)$ with domain A such that
   $$(\forall x \in A)\mathcal{P}(x, F(x), A, x_1, \ldots, x_k).$$

Idealization:

For any $\mathbf{V}(\alpha) \subset \mathbf{V}(\beta)$, any $A \in \mathbf{V}(\alpha)$, and any $x_1, \ldots, x_k$,

$$(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{V}(\alpha))(\exists y)(\forall x \in a)\mathcal{P}^{\beta}(x, y, A, x_1, \ldots, x_k)$$
$$\iff (\exists y)(\forall x \in A \cap \mathbf{V}(\alpha))\mathcal{P}^{\beta}(x, y, A, x_1, \ldots, x_k).$$

Corollary:

If $\mathbf{V}(\alpha) \subset \mathbf{V}(\beta)$, then there are natural numbers n such that

$$n \in \mathbf{V}(\beta)\,,\; n \notin \mathbf{V}(\alpha)\,.$$

Local Transfer Principle.

Let $\mathcal{P}(x_1, \ldots, x_k)$ be any $\in$-$\sqsubseteq$-formula.
If $\mathcal{P}^\alpha(x_1, \ldots, x_k)$ holds, then there exists $\gamma \sqsupset \alpha$ such that $\mathcal{P}^\beta(x_1, \ldots, x_k)$ holds for all β with $\alpha \sqsubseteq \beta \sqsubset \gamma$.

The point is that $x_1, \ldots, x_k$ are arbitrary; they do not have to belong to $\mathbf{V}(\alpha)$!

Axioms of GRIST:

We strengthen Idealization and Standardization, and add Granularity.

Idealization:

For any $\mathbf{V}(\alpha) \subset \mathbf{V}(\beta)$, any $A \in \mathbf{V}(\alpha)$, and any $x_1, \ldots, x_k$,

$$\begin{aligned} &(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{V}(\alpha))(\exists y)(\forall x \in a)\mathcal{P}^{\beta}(x, y, A, x_1, \ldots, x_k) \\ \iff &(\exists y)(\forall x \in A)[\mathbf{V}(x) \subset \mathbf{V}(\beta) \Rightarrow \mathcal{P}^{\beta}(x, y, A, x_1, \ldots, x_k)]. \end{aligned}$$

Standardization:

Given A such that $\mathbf{V}(0) \subset \mathbf{V}(A)$, and any $x_1, \ldots, x_k$, there exists B such that $\mathbf{V}(B) \subset \mathbf{V}(A)$ and, for every β with $\mathbf{V}(B) \subseteq \mathbf{V}(\beta) \subset \mathbf{V}(A)$ and every $z \in \mathbf{V}(\beta)$,

$$z \in B \iff z \in A \ \wedge \ \mathcal{P}^{\beta}(z, A, x_1, \ldots, x_k)).$$

Granularity:

For any $x_1, \ldots, x_k$, if $(\exists\alpha)\mathcal{P}^\alpha(x_1, \ldots, x_k)$, then

$$(\exists\alpha)[\mathcal{P}^\alpha(x_1, \ldots, x_k) \wedge (\forall\beta)(\mathbf{V}(\beta) \subset \mathbf{V}(\alpha) \Rightarrow \neg\, \mathcal{P}^\beta(x_1, \ldots, x_k))].$$

Theorem 1. ***GRIST*** *has an interpretation in* ***ZFC****, in which* $\mathsf{V}(0)$ *is isomorphic to the universe* $\mathbf{V}$ *of sets of* ***ZFC****.*

The interpretation is given by a complicated limit ultrapower.

Corollary 2. ***GRIST*** *is a conservative extension of* ***ZFC****. In particular,* ***GRIST*** *is consistent relative to* ***ZFC****.*

Corollary 3. *Every model* $\mathfrak{M}$ *of* ***ZFC*** *has an extension to a model* $\mathfrak{N}$ *of* ***GRIST*** *(where sets of* $\mathfrak{M}$ *are precisely the sets of* $\mathfrak{N}$ *that appear at the level* $\mathsf{V}(0)$*).*

Theorem 4. ***(Reduction Algorithm)***
There is a formula $x\mathbb{M}_\alpha U$ of the $\in$-$\sqsubseteq$-language and a formula $\mathbb{S}(U)$ of the $\in$-language such that

$$(\forall x)(\exists U)[\,\mathbb{S}(U) \ \wedge\ U \in \mathbf{V}(\alpha) \ \wedge\ x\mathbb{M}_\alpha U\,];$$
$$(\forall U)[(\mathbb{S}(U) \ \wedge\ U \in \mathbf{V}(\alpha)) \to (\exists x)\ x\mathbb{M}_\alpha U].$$

Moreover, for every formula $\mathcal{P}(x_1, \ldots, x_k)$ of the $\in$-$\sqsubseteq$-language there is a formula $\mathcal{Q}(U)$ of the $\in$-language (effectively obtained from it) such that

$$\langle x_1, \ldots, x_k\rangle\mathbb{M}_\alpha U \Rightarrow (\mathcal{P}^\alpha(x_1, \ldots, x_k) \iff \mathcal{Q}(U)).$$

Corollary 5. *If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two extensions of a model $\mathfrak{M}$ of **ZFC** to a model of **GRIST**, then they are $L_{\infty,\omega}$-elementarily equivalent.*

Corollary 6. *Every countable model $\mathfrak{M}$ of **ZFC** has a* unique *(up to isomorphism which is identity on $\mathfrak{M}$) extension to a countable model of **GRIST**.*

Corollary 7. *If $\mathcal{T}$ is a complete consistent extension of **ZFC** (in $\in$-language), then $\mathcal{T}$+**GRIST** is a complete consistent theory in the $\in$-$\sqsubseteq$-language.*

Corollary 8. ***GRIST** is finitely axiomatizable over **ZFC**.*

Corollary 9. *If x is uniquely definable in **GRIST** from parameters in $\mathbf{V}(0)$, then x belongs to $\mathbf{V}(0)$. If $x \notin \mathbf{V}(0)$, then for each α there exist $y \in \mathbf{V}(\alpha)$, $y \notin \mathbf{V}(\beta)$ for any $\beta \sqsubset \alpha$, such that x and y are $\in$-$\sqsubseteq$-indiscernible.*

Let f be a function and a a real number.
If f is differentiable at a, then, for each dx supersmall relative to the level of f and a,

$$f(a + dx) = f(a) + f'(a) \cdot dx + \varepsilon \cdot dx,$$

for some $\varepsilon \simeq 0$ (relative to the level of f and a).

A **tagged partition** of $[a; b]$ is a finite set $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 < x_1 < \ldots < x_i < \ldots < x_n = b.$$

and a set $\mathcal{T} = \{t_0, \ldots, t_{n-1}\}$ where

$$x_i \leq t_i \leq x_{i+1}, \qquad \text{for } i = 0, \ldots, n-1.$$

We let

$$dx_i = x_{i+1} - x_i$$

.

A partition is **fine** if all dx_i are supersmall relative to the context level.

The function f is **Riemann integrable** on $[a, b]$ if there is $R \in \mathbb{R}$ in $\mathbf{V}(f, a, b)$ such that

$$\sum_{i=0}^{n-1} f(t_i) \cdot dx_i \simeq R,$$

for all fine tagged partitions $\mathcal{P}, \mathcal{T}$ of $[a; b]$.

Definition 10. Given a context level and a real number a:

1. A real number r is a-**accessible** if $r = \varphi(a)$ for some function $\varphi : \mathbb{R} \to \mathbb{R}$ in the context level.
2. A real number $h \neq 0$ is a-**supersmall** if $|h| \leq r$ for all a-accessible $r > 0$.

We say that a function φ is **positive** if $\varphi(x) > 0$ for all x in its domain. It follows immediately from these definitions that $h \neq 0$ is a-supersmall if and only if $|h| \leq \varphi(a)$ for all positive $\varphi : \mathbb{R} \to \mathbb{R}$ in the context level.

Theorem 11. *The following statements are equivalent:*

1. $\lim_{x \to a} f(x) = L$
2. *The number L is a-accessible relative to $\mathbf{V}(f)$ and $f(a+h) - L$ is a-supersmall relative to $\mathbf{V}(f)$ (or 0), for all h a-supersmall relative to $\mathbf{V}(f)$.*

Definition 12. A tagged partition $(\mathcal{P}, \mathcal{T})$ is **superfine** relative to the level $\mathbf{V}(\alpha)$ if each dx_i is t_i-supersmall relative to $\mathbf{V}(\alpha)$.

Theorem 13. *If $(\mathcal{P}, \mathcal{T})$ is a superfine partition of $[a; b]$, then every real number $c \in [a; b]$ appearing at the context level belongs to $\mathcal{T}$.*

Theorem 14. *Let $a, b \in \mathbb{R}$ and let $\{I_k\}_{k=1}^{\infty}$ be a system of open intervals appearing at the context level. If $(\mathcal{P}, \mathcal{T})$ is a superfine partition of $[a; b]$, then for each $t_i \in \bigcup_{k=1}^{\infty} I_k$ there is some k such that $[x_i; x_{i+1}] \subseteq I_k$.*

Definition 15. A function f defined on $[a;b]$ is **generalized Riemann integrable on** $[a;b]$ if there is a number R appearing at the context level such that

$$\sum_{i=0}^{n-1} f(t_i) \cdot dx_i \simeq R,$$

for all superfine tagged partitions $(\mathcal{P}, \mathcal{T})$ of $[a;b]$.