Analysis using Relative Infinitesimals

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Axiomatic properties of levels

1. Real numbers are stratified in levels.
2. There is a coarsest level and 1 appears at this level.
3. For each level, there are numbers (even integers) which do not appear at this level (we say that they appear at *finer* levels).
4. If a number appears at a given level, it appears at all finer levels.

Definitions

Given a level:
A real number h is ultrasmall relative to this level if $h \neq 0$ and $|h| < c$ for any positive c appearing at this level.

A real number N is ultralarge relative to this level if $|N| > c$ for any positive c appearing at this level.

Two real numbers a and b are ultraclose relative to this level if their difference is either ultrasmall or zero.
This is written

$$a \simeq b.$$

Axioms for elementary teaching

1. Each real number appears at a level.
2. The number 1 appears at the coarsest level.
3. If a number appears at a given level, it also appears at all finer levels.
4. At the coarsest level there appear no ultrasmall nor ultralarge numbers.
5. For each level, there are ultrasmall and ultralarge numbers relative to that level.

Example of level

The level of the function

$$f : x \mapsto ax^2 + bx + c$$

is the level of a, b and c.

The level of $f(x)$ is the level of a, b, c and x.

Definition: Context level

The **context level** of a property is the coarsest level at which appear all parameters needed to specify it.

Example of context level

“The equation $ax^2 + bx + c = 1$ has a solution“

The context level of this statement is the level of a, b, c.

Axiom: Closure Principle

If a property which does not mention levels is true for some number, then it is true for a number appearing at the context level.

Example of closure

“The equation $ax^2 + bx + c = 1$ has a solution“

If the equation has a solution, then it has a solution appearing at the level of a, b and c.

Application of the closure principle

Theorem: Given a level:
If a and b appear at that level then

$$a \simeq b \implies a = b.$$

Proof: Relative to the level, $a \simeq b$ implies that $b - a$ is ultrasmall or $b - a = 0$.

By closure, $b - a$ appears at the level. So $b - a$ is not ultrasmall hence $b - a = 0$ and $a = b$. □

Definition: Continuity of f at a

Continuity of f at a is a property of f and a, hence the context level will be the level of f and a.

$|N| > c$. Let f be a real function defined around a.
We say that f is continuous at a if, for all x,

$$x \simeq a \implies f(x) \simeq f(a).$$

Example of continuity

Claim: $f : x \mapsto x^2$ is continuous at a.
The context level is the level of a (f appears at the coarsest level).
Let h be ultrasmall.

$$f(a+h) = a^2 + 2a \cdot h + h^2 \simeq a^2 = f(a).$$

Axiom: Transfer Principle (the case of continuity)

The following properties are equivalent.

(a) Relative to the level of f and a:
For each x, if $x \simeq a$ then $f(x) \simeq f(a)$.

(b) Relative to a finer level:
For each x, if $x \simeq a$ then $f(x) \simeq f(a)$.

This is what allows us to work relative to a context level:

It doesn't matter what the context level is,
provided it is sufficiently fine.

Application of the transfer principle

Theorem: If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Proof: The context level is given by f, g and a.
By transfer we can use this level in the definition of continuity of g at a and f at $g(a)$.

Let $x \simeq a$.

$$x \simeq a \implies g(x) \simeq g(a) \implies f(g(x)) \simeq f(g(a)).$$

□

Axiom: Neighbour Principle

Given a level:
If a number is not ultralarge then it is ultraclose to a real number appearing at the level.

If the level is the context level, we say it is the context neighbour.

Application of the neighbour principle

Consider the function

$$x \mapsto x^2$$

at a. The context level is the level of a.

Let h be ultrasmall.
Then the context neighbour of

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2-a^2}{h} = 2a+h$$

is $2a$.

Application: Intermediate Value Theorem

Let f be a real function continuous on $[a; b]$. Let d be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a; b]$ such that $f(c) = d$.

(wlog assume $f(a) < d < f(b)$)

The context level is the level of f, a, b and d.

proof

Let N be a ultralarge positive integer.
Partition the interval $[a; b]$ into N even parts of length $\frac{b-a}{N}$:

$$a = x_0, \; x_1, \; \ldots, \; x_N = b.$$

Let j be the first integer such that $f(x_j) \geq d$, hence $f(x_{j-1}) < d$.
Let c be the context neighbour of x_j (and x_{j-1}).

By continuity of f at c we have

$$f(x_{j-1}) \simeq f(c) \quad \text{and} \quad f(c) \simeq f(x_j).$$

Hence $f(c) \simeq d$.

By closure, $f(c)$ appears at the context level. As d also appears at the contet level we have $f(c) = d$.

□