On Hindman Sets

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My talk is about possible extensions of the following well-known theorem:

Theorem (Hindman)**.** *For any partition of* $\mathbb{N}$ *into finitely many pieces there exist a piece* H *and an infinite subset* $B \subseteq H$ *such that* H *contains all the finite sums of distinct elements of* B*.*

This was proved in 1974 and sharped early results of many authors starting from Hilbert. The theorem can be extended to semigroups:

Theorem. *Let* $(X, \cdot)$ *be a right cancellative semigroup. For any partition of* X *into finitely many pieces there exist a piece* H *and a sequence* $(x_i)_{i<\omega}$ *of distinct elements of* H *such that*

$$x_{i_{n+1}} x_{i_n} \cdots x_{i_2} x_{i_1} x_{i_0} \in H$$

whenever $i_0 < i_1 < i_2 < \cdots < i_n < i_{n+1}$ *and* $n < \omega$*.*

For further generalizations, let me introduce the following terminology:

Let $(X,\cdot)$ be a groupoid (a set with a binary operation, not necessary associative). A set $H \subseteq X$ is a *Hindman* set w.r.t. a sequence $(x_\alpha)_{\alpha<\kappa}$ of distinct elements of H if

$$((\cdots(x_{\alpha_{n+1}}x_{\alpha_n})\cdots x_{\alpha_2})x_{\alpha_1})x_{\alpha_0} \in H$$

whenever $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1}$ and $n < \omega$. A κ-*Hindman* set is a Hindman set w.r.t. some κ-sequence.

Thus Hindman's theorem states that, if the operation is associative and right cancellative, then any finite partition of X contains some ω-Hindman piece.

Mainly, my talk is on existence of Hindman sets in *infinite partitions* and/or w.r.t. sequences of *transfinite length*. A minor part concerns Hindman sets for multiple structures and non-associative operations.

Hindman's theorem has close relationships with idempotent ultrafilters: the simplest way (discovered by Galvin and Glazer) to get it is to prove two following results:

Lemma A. *Any compact left topological semigroup contains an idempotent.*

To formulate the second, recall that any binary operation on a set S can be uniquely extended to a left continuous binary operation on ßS, the space of all ultrafilters on S. (Left continuity means here that the operation is continuous whenever the 1st variable is fixed. We denote the extended operation by the same symbol.)

Lemma B. *Let* $(S, \cdot)$ *be a discrete semigroup and* $U \in$ ß$S \setminus S$ *an idempotent ultrafilter. Then every* $A \in U$ *is* ω*-Hindman.*

To get results on infinite partitions, we follow the same strategy but deal with infinitely additive (ultra)filters.

First, we extend Lemma A:

Let X be a topological space. Let us say that *quasicharacter* of an $S \subseteq X$ is greather than λ if the intersection of λ neighborhoods of S includes some one.

Example. Let $S \subseteq ßX$ be closed and so $S = \overline{D}$ for some filter D on X (where $\overline{D} = \{U \in ßX : D \subseteq U\}$). Then quasicharacter of S is add (D) while character of S is cof (D).

Lemma. *Let $(X, \cdot)$ be a compact left topological semigroup such that any $S \subseteq X$ of quasicharacter $> \lambda$ contains an element $x \in S$ of quasicharacter $> \lambda$. Then X contains an idempotent of quasicharacter $> \lambda$.*

Proof (scetch). We follows the standard proof as near as possible:

To find a minimal compact subsemigroup S of quasicharacter $> \lambda$, we apply Zorn's lemma (since the intersection of any chain of compact subsemigroups of quasicharacter $> \lambda$ is such a subsemigroup).

We then take $e \in S$ of quasicharacter $> \lambda$ and show that $eS = S$ and $\{x \in S : ex = e\} = S$. (Besides usual arguments, we have to check that both eS and $\{x \in S : ex = e\}$ have quasicharacter $> \lambda$.) We conclude that $S = \{e\}$. $\square$

Recall that a cardinal κ is *strongly compact* if, whenever $|X| \geq \kappa$, any κ-additive filter on X can be extended to some κ-additive ultrafilter on X.

Corollary. *Let κ be a strongly compact cardinal and $(X, \cdot)$ a semigroup with $|X| \geq \kappa$. Then $(ßX, \cdot)$ contains an idempotent which is κ-additive.*

Now we extend Lemma B:

Lemma. *Let* $(X, \cdot)$ *be a groupoid and* D *a non-principal filter on* X *such that* $\overline{D}$ *is a sub-groupoid of* $(ßX, \cdot)$. *Then any* $A \in D$ *is* κ*-Hindman with* $\kappa = \mathrm{add}\,(D)$.

Proof. An appropriate modification of the standard proof. At limit steps, we use additivity and take intersections. □

Corollary. *Let* $(X, \cdot)$ *be a groupoid. If there exists* $U \in ßX \setminus X$ *which is an idempotent of* $(ßX, \cdot)$ *and* $\kappa = \mathrm{add}\,(U)$, *then any partition of* X *into* $< \kappa$ *pieces contains a* κ*-Hindman piece.*

(Notice that these claims require no associativity or other properties of the operation.)

Putting all together, we get

Theorem. *Let κ be a strongly compact cardinal and $(X, \cdot)$ a right cancellative semigroup with $|X| \geq \kappa$. Then any partition of X into $< \kappa$ pieces contains a κ-Hindman piece.*

Question. Characterize cardinals κ such that, whenever $(X, \cdot)$ a right cancellative semigroup with $|X| \geq \kappa$, then any partition of X into $< \kappa$ pieces contains a κ-Hindman piece.
The existence of κ-additive idempotents is sufficient but not obviously necessary. My guess is that such cardinals are weakly compact (or may be Ramsey, or something like).

Multiple structures.

Since Hindman's theorem is applicable to both semigroups $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$, one can ask: Is it possible to find a piece which is

(i) additively and multiplicatively Hindman simultaneosly? and
(ii) additively and multiplicatively Hindman w.r.t. the same sequence?

The answer to the first is positive:

Theorem (Bergelson–Hindman). *For any partition of $\mathbb{N}$ into finitely many pieces there exist a piece H and infinite $B, C \subseteq H$ such that H contains all the finite sums of distinct elements of B and all the finite products of distinct elements of C.*

More recently Bergelson proved some sharper results.

A heavy Hindman's result answers to the second negatively:

There exists a finite partition of $\mathbb{N}$ in which no piece is additively and multiplicatively Hindman w.r.t. the same ω-sequence.

Another negative result follows as a corollary: no $U \in \mathbb{N}^*$ satisfies $U + U = U \cdot U$.

Multiple version of A:

Let us call $(X, +, \cdot)$ a (left topological) semiring if both $(X, +)$ and $(X, \cdot)$ are (left topological) semigroups and multiplication is left distributive w.r.t. addition.

Lemma. *Let $(X, +, \cdot)$ be a compact left topological semiring such that any $S \subseteq X$ of quasicharacter $> \lambda$ contains an element $x \in S$ of quasicharacter $> \lambda$. Then X contains a common idempotent $e = e + e = e \cdot e$ of quasicharacter $> \lambda$.*

Since $(\mathbb{N}^*, +, \cdot)$ contains no common idempotents, we see that no its closed subsemiring satisfies left distributivity.

(A version of this lemma for many operations.)

Question. That any minimal compact left topological semigroup consists of a single element is provable in ZF alone. Unlike this, our proof of the fact that any minimal compact left topological semiring consists of a single element uses Zorn's lemma. Is AC really necessary for the semiring version?

Multiple version of B:

Lemma. *Let* $(X, (F_\alpha)_{\alpha<\xi})$ *be a universal algebra with binary operations and* $E_\alpha = \{U \in ßX : F_\alpha(U,U) = U\}$.
(i) If there exists

$$U \in \bigcap_{\alpha<\xi} E_\alpha$$

with $\kappa = \mathrm{add}\,(U)$, *then any partition of* X *into* $< \kappa$ *pieces contains a piece which is Hindman for all* F_α*'s w.r.t. the same* κ*-sequence.*
(ii) If there exists

$$U \in \bigcap_{\alpha<\xi} \mathrm{cl}\,(E_\alpha)$$

with $\kappa = \mathrm{add}\,(U)$, *then any partition of* X *into* $< \kappa$ *pieces contains a piece which is* κ*-Hindman for all* F_α*'s.*

Putting together, we get

Theorem. *Let* κ *be a strongly compact cardinal and* $(X, +, \cdot)$ *a right cancellative semiring with* $|X| \geq \kappa$*. Then any partition of* X *into* $< \kappa$ *pieces contains a piece which is additively and multiplicatively* κ*-Hindman.*

Relationships to AC. Partitions of ω_1.

Hindman's theorem for the case $\mathbb{N}$ is provable in ZF alone (since arithmetic is absolute).

Question. Is there a conterexample to Hindman's theorem in ZF? (Probably, yes.)

Let $\alpha + \beta$ be the usual addition of ordinals, and let $\alpha \dot{+} \beta = \beta + \alpha$. Notice that $\dot{+}$ is right cancellative.

Remark. The usual ordinal addition (which left but not right cancellative) gives a simple example when Hindman's theorem fails for not right cancellative semigroups.

In ZFC, the club filter on ω_1 is countably additive, and so we have:

Claim. *Every countable partition of* $(\omega_1, \dot{+})$*, or even of any its stationary subsemigroup, contains an* ω_1*-Hindman piece.*

Proof. An obvious club sequence $(\alpha_\xi)_{\xi<\omega_1}$ satisfies $\alpha_{\xi_1} \dot{+} \alpha_{\xi_0} = \alpha_{\xi_1}$ whenever $\xi_0 < \xi_1$. $\square$

Certainly, the claim holds not only for usual ordinal addition, but also for ordinal multiplication, exponentiation, etc. (no associativity requires).

Notice a difference between ω and ω_1: the latter contains common idempotents.

This claim is not provable in ZF alone: simply put $\mathrm{cf}\,\omega_1 = \omega$. Then one can ask:

Question. Is some of the following sentences provable in ZF:
(i) Any countable partition of $(\omega_1, \dot{+})$ contains an ω-Hindman piece?
(ii) Any finite partition of $(\omega_1, \dot{+})$ contains an ω_1-Hindman piece?

On the other hand, Determinacy proves the stronger claim:

Claim. *Assume* AD. *Let* $(\omega_1, \cdot)$ *be arbitrary semigroup. Then every countable partition of* ω_1 *contains an* ω_1*-Hindman piece.*

Proof. Under AD, the club filter on ω_1 is an idempotent ultrafilter. □

Let me conclude my talk with two questions:

Question. Assume AD. Let $\mathbb{R}$ be partitioned into $< \Theta$ pieces. Is there a piece H which is Hindman w.r.t. a perfect set (i.e. contains a perfect subset such that all the finite sums of its distinct elements are in H)?

Question. Let $\mathbb{N}$ be partitioned into finitely many pieces. Is there a piece H which is Hindman for addition, multiplication, and exponentiation (and further obvious primitive recursive operation)?