Nonstandard methods in algebraic geometry

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- joint work with Lars Brünjes from Regensburg, Germany

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- slides of this talk and our articles will soon be available on my homepage wwwmath.uni-muenster.de/u/serpe/

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- in the characteristic zero case one can use transcendental methods,
- and in characteristic $p > 0$ case one has the Frobenius morphism.

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A link between the apparently so different worlds might be provided by the ultra product

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of the finite fields $\mathbb{F}_p$ where M is an infinite set of primes and $\mathcal{U}$ is an ultra filter.

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- $char(\prod_{p \in M, \mathcal{U}} \mathbb{F}_p) = 0$
- in some sense $\prod_{p \in M, \mathcal{U}} \mathbb{F}_p$ behaves like a finite field.

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A **morphism** of two algebraic subsets/affine varieties is a map which is given by polynomials. So we have the category **AffVar/K.**

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This gives a link between varieties over fields of characteristic zero and varieties over fields of charactereistic $p > 0$.

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What is ${}^{*}Sch^{fp}/K$?

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- here: $\{Sch^{fp}/k\}_{k \in S} \rightsquigarrow \{{}^*Sch^{fp}/K\}_{K \in {}^*S}$

Construction of N

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- find a subring $A_0 \subset {}^*\mathbb{F}_P$ of finite type over $\mathbb{Z}$ and a scheme $X_0 \in Sch^{fp}/A_0$ such that $X = X_0 \otimes_{A_0} {}^*\mathbb{F}_P$

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For schemes X, Y over an internal field we have:

- X is a variety if and only if $N(X)$ is a *variety (uses a result of van den Dries/Schmidt about the map $K[x_1, \dots, x_n] \to K^*[x_1, \dots, x_n]$)

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- $f : X \to Y$ is birational if and only if $N(f) : N(X) \to N(Y)$ is *birational

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There is a cofinite set of primes $\mathbb{P}' \subset \mathbb{P}$ such that for all schemes X over a field of characteristic $p \in \mathbb{P}'$ with $X \in S$ the statement Φ holds.

Examples

Theorem (Eklof 69)

For any pair (n, d) of natural numbers, there exists a bound $C \in \mathbb{N}$ such that for any field of characteristic $p > C$ and any closed subvariety X of $\mathbb{P}_k^n$ of degree d, there exists a resolution of singularities of X.

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A similar results holds for weak factorization

Étale cohomology and algebraic cycles

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And there is a **cycle class map**

$$cl : Z^i(X) \to H^{2i}_{et}(X, \mathbb{Z}/m)$$

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Proposition (B.-S.)

It is possible to construct a canonical morphisms

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Let X be a proper scheme over an internal separably closed field. Then the canonical morphism

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For cycles the map N is far from being surjective.

Lifting divisors to characteristic zero

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