105AA Hyperbolic dynamics (Dinamica Iperbolica) Second semester 2023-24 6 CFU, 42 hours

Instructor: Gianluigi Del Magno

1 Overview

This course provides an introduction to hyperbolic dynamics, a very active area of dynamical systems. A hyperbolic system refers to a smooth transformation on a differentiable manifold that is characterized, in broad terms, by the simultaneous presence of expanding and contracting directions for the derivative of the transformation.

Hyperbolicity stands as the primary mechanism responsible for generating chaotic behavior in dynamical systems. Although the time evolution of a hyperbolic system is deterministic, its behavior resembles, in many ways, a stochastic process. Key indicators of hyperbolic behavior include the coexistence of dense and periodic orbits, sensitivity to initial conditions, and exponential growth in the number of periodic orbits with respect to their periods. Another notable characteristic of hyperbolicity is that it persists under small perturbations of the dynamics. The best-known example of a hyperbolic system is Smale's horseshoe, which abstracts the stretch-and-fold mechanism associated with hyperbolic behavior.

Hyperbolicity was discovered by Poincaré in his renowned work on the three-body problem, where he showed that in a neighborhood of a transverse homoclinic point[1] a system exhibits hyperbolic behavior. Important classes of hyperbolic systems related to geometry and statistical mechanics are geodesic flows on Riemannian manifolds with negative sectional curvature and hyperbolic billiards[2].

[1] https://en.wikipedia.org/wiki/Homoclinic_orbit

[2] A billiard can be thought of as a geodesic flow on a manifold with boundary and corners.

Below are links to beautiful videos on hyperbolic flows, hyperbolic billiards and Smale horseshoe:

1. Hyperbolic geodesic flows and billiards, Chaos, Episode 5 by Leys, Ghys and Alvarez,
   https://www.youtube.com/watch?v=lnAanzNT0Gk&list=PLw2BeOjATqrv6B0vq-rgsfa1G0KuGdLKO&index=5

2. Smale horseshoe, Chaos, Episode 6 by Leys, Ghys and Alvarez,
   https://www.youtube.com/watch?v=QNFzceZYWlU&list=PLw2BeOjATqrv6B0vq-rgsfa1G0KuGdLKO&index=6

2 Course Description

This 42-hour course consists of three parts divided into 21 lectures. The first part is devoted to the study of approachable examples of hyperbolic systems illustrating the important features of the theory: hyperbolic toral automorphisms, Smale horseshoe and Smale solenoid. The second part covers the main results of the general theory, such as persistence of hyperbolicity, stable manifold theory, shadowing and the spectral decomposition theorem. The third part concerns more advanced topics to be selected according to the interests of the students and the instructor. Examples of such topics are hyperbolic attractors, physical measures, ergodicity and Hopf's argument, hyperbolic flows and hyperbolic billiards.

3 Course Outline

1. Basic concepts of dynamical systems
2. Hyperbolic sets
3. Examples of hyperbolic systems: Smale horseshoe, solenoid, toral automorphisms
4. Persistence of hyperbolicity
5. Stable manifold theorem
6. Shadowing
7. Spectral decomposition theorem

8. Advanced topics to be selected from hyperbolic flows, hyperbolic billiards, physical measures for hyperbolic attractors

4 Bibliography

1. L. Wen, Differentiable dynamical systems. An introduction to structural stability and hyperbolicity, Graduate Studies in Mathematics 173, AMS, 2016 (the e-book can be downloaded freely from the university library)

2. A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, 1995

3. M. Brin and G. Stuck, Introduction to dynamical systems, Cambridge University Press, 2002

4. L. Barreira and C. Valls, Dynamical systems: an introduction, Springer-Verlag London 2013

5. Research articles on hyperbolic dynamics

5 Assessment

40-minute seminar on a topic agreed upon by the student and the instructor followed by a 15-minute examination on the general theory presented in the course. A list of topics for the seminar will be provided by the instructor.

6 Prerequisites

Real Analysis, Linear Algebra, Basic notions of dynamical systems.

7 Instructor

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