Course Description
Differential Geometry and Topology
(Geometria e Topologia Differenziale)

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Introduction

The course is divided into two parts, of 40 and 20 hours respectively. The first part covers the basics of the differential geometry of curves and surfaces in three-dimensional space. The second part is an introduction to differential topology.

First Part Highlights

- *Curvature and torsion:* these two quantities are associated to a space curve. They vary along the curve and describe the way the curve bends and twists in space.
- *Shape operator:* it is associated to each point p of a surface in 3-space. It is a self-adjoint operator defined on the tangent plane at p. Around a point a generic surface looks like the graph of a real function $f(x, y)$ with an isolated, non-degenerate critical point at the origin. The eigevalues of the shape operator determine whether f has a local maximum, minimum or a saddle point.
- *Gaussian curvature:* it is the determinant of the shape operator. For example, the Gaussian curvature of a standard round sphere is positive at every point while that of a standard plane vanishes everywhere.
- *Gauss's Theorema Egregium:* the Gaussian curvature is determined by angles and distances on a surface regardless of the way the surface is embedded in 3-space. This theorem implies that it is impossible to draw a planar map of any portion of the earth preserving path lengths.
- *Geodesics:* these are special curves on a surface which locally minimize the distance between any two of their points.
- *Theorem of Gauss-Bonnet:* the integral of the Gaussian curvature over the surface can be expressed in terms of the *Euler characteristic*, which only depends on the global topology of the surface.

Second Part Highlights

- *Smooth manifolds:* these can be defined as specific subsets of Euclidean spaces. They are endowed with a dimension and tangent spaces. For instance, the orthogonal group $O(n)$ is a smooth manifold of dimension $n(n-1)/2$. There is a well-defined notion of a smooth map between smooth manifolds.

- *Degree modulo* 2*:* it is a quantity associated to a map f between two compact, smooth manifolds. The degree modulo 2 is unchanged by smooth homotopies of f. It can be used to prove Brower's theorem.

- *Brower's theorem:* states that a continuous map from the n-dimensional ball to itself must have a fixed point.

- *Degree:* it is the analogue of the degree modulo 2 for smooth maps between smooth, compact and *oriented* manifolds.

- *Hairy ball theorem:* says that there is no smooth tangent vector field on an even-dimensional sphere.

- *Index of a smooth vector field at an isolated zero:* it is the degree of the map obtained by restricting the normalization of the vector field to the boundary of a small ball centered at the isolated zero.

- *Poincaré-Hopf theorem:* it generalizes the hairy ball theorem. Given a smooth vector field v with isolated zeros on a smooth, compact manifold M, the sum of the indices of the zeros of v is equal to the *Euler characteristic* $\chi(M)$, which only depends on the global topology of M.

Additional information

- *Prerequisites:* calculus in one and several variables and some general topology.

- *Exam structure:* a written test on the first part and an oral examination on the second part. There will be a total of 6 exam sessions.

- *Textbooks:*

  (i) M. P. Do Carmo, "Differential Geometry of Curves and Surfaces", Courier Dover Publications, 2016.

  (ii) V. Guillemin, A. Pollack, "Differential Topology", AMS Chelsea Publishing, 1974.

  (iii) J. Milnor, "Topology from the Differentiable Viewpoint", UP of Virginia, Charlottesville, 1965.

  (iv) T. Shifrin, "Differential Geometry: A First Course in Curves and Surfaces", available at
  http://alpha.math.uga.edu/ shifrin/ShifrinDiffGeo.pdf