Geometria Riemanniana

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1 Outline

Riemannian geometry is a vast subject encompassing diverse aspects. In the course, I will focus on three main topics.

Jacobi fields and their applications

Jacobi fields are vector fields along a geodesic, which arise by considering geodesic variations, i.e. variations of a geodesic through geodesics. They can be used to express the differential of the exponential map at a point and the Hessian of the square distance from a given point. These geometric properties of Jacobi fields are used in the proof of certain theorems which put constraints on the topology of a Riemannian manifold satisfying special curvature conditions, much in the same spirit as the classical Gauss-Bonnet theorem for surfaces.

I will cover the celebrated theorems of Myers and Cartan-Hadamard, as well as two theorems involving the injectivity radius, namely Klingerberg's theorem on the injectivity radius of a metric with bounded positive sectional curvature and Berger's sphere theorem.

This part of the course will follow Chapter 6 of [2].

The isometry group

There are two notions of isometry one can give on a Riemannian manifold: as metric spaces, or as smooth maps that preserve the scalar product. By a theorem of Myers-Steenrod, the two definitions turn out to be equivalent. Even less obvious is the fact (also due to Myers-Steenrod) that the group of isometries is a manifold, and more precisely, a Lie group.

I will give a proof of this fact using the language of moving frames, as given in Chapter 2 of [1]. This approach is mostly useful for generalizing from the isometry group to groups that preserve a G-structure, but even if there will be no time to investigate this, introducing the language of principal bundles, somewhat ubiquitous in differential geometry, has its own merit.

I will also show that the group of isometries has dimension at most $\frac{1}{2}n(n+1)$, and maximal dimension is achieved by spaces of constant curvature. This ties in with the last topic of the course.

Symmetric spaces

A Riemannian manifold M is locally symmetric if the curvature tensor is parallel; spaces of constant curvature are a special case. This condition implies that around each point x there is an isometry fixing x with differential $-\mathrm{id}\colon T_xM \to T_xM$. If one further assumes M is complete and simply connected, the symmetry extends to a global isometry; one then says that M is globally symmetric, or simply a symmetric space.

Symmetric spaces are in particular homogeneous spaces, i.e. they take the form G/H, where G is a Lie group and H a closed subgroup; accordingly, they are studied in terms of the corresponding Lie algebras.

I will give some structural results, in addition to some results on the curvature, fundamental group, and isometry group of a Riemannian symmetric space.

For this part of the course, I will refer to Chapter 8 of Wolf's book [3]. Notice that chapter and book go far beyond what I will be able to cover in the course.

2 Practical information

This course takes off where "Istituzioni di Geometria" ends. It is advisable to follow "Geometria e Topologia differenziale" first, then "Istituzioni di Geometria", before taking Riemannian geometry.

Failing that, the students will have to fill in some prerequisites on their own. The following will be assumed as prior knowledge:

- smooth manifolds and fiber bundles;
- Riemannian metrics; Levi-Civita-connection, curvature;
- vector fields; Lie derivative; flow of a vector field
- geodesics and exponential map;
- basics of Lie algebras and Lie groups. Notice that I will *not* assume that the audience is familiar with roots, Dynkin diagrams and the classification of simple Lie algebras.

I do not expect to deviate from the indicated bibliography. However, I plan to write lecture notes as I go, to expand on the details as needed.

References

[1] S. Kobayashi, Transformation groups in differential geometry. Springer.

[2] Petersen, Riemannian Geometry

[3] Wolf. Spaces of constant curvature. AMS Chelsea Publishing.