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1. Cardinal arithmetic

1.1. **The cofinality of cadinals.** The general definition of cof A as the smallest order-type of a well-ordered cofinal subset of the ordered set A, when applied to cardinals has the following useful characterization:

$$\operatorname{cof} \kappa = \min \{|I| \mid \kappa = \sum_{i \in I} \kappa_i, \quad with\ \kappa_i < \kappa\ \forall i \in I\}.$$

Recall that a cardinal is regular if $\operatorname{cof} \kappa = \kappa$ and singular otherwise. It follows that all successor cardinals are regular, because

$$\sum_{i \in I} \kappa_i = \max\,(|I|, \sup_I \kappa_i).$$

On the other hand, $\operatorname{cof} \aleph_\alpha = \operatorname{cof} \alpha \leq \alpha$, so most limit cardinals are singular. A regular limit cardinal is called weakly inaccessible.

1.2. **König-Zermelo inequality.** The basic strict inequality among cardinals is the König-Zermelo inequality

$$\forall i \in I\ \kappa_i < \nu_i \implies \sum_{i \in I} \kappa_i < \prod_{i \in I} \nu_i,$$

that actually implies all known strict inequalities in cardinal arithmetic.

E.g. Cantor's Theorem $2^{|I|} > |I|$ follows by putting $\kappa_i = 1$ and $\nu_i = 2$.

More generally one obtains $\operatorname{cof}\ \kappa^\nu > \nu$, namely, for $\alpha < \nu$ let $\kappa_\alpha < \kappa^\nu$; then

$$\sum_{\alpha < \nu} \kappa_\alpha < \prod_{\alpha < \nu} \kappa^\nu = (\kappa^\nu)^\nu = \kappa^\nu.$$

1.3. **The power in base 2.** The cardinal $2^\kappa = |\mathcal{P}(\kappa)|$ satisfies the monotonicitiy condition

$$(d1) \qquad \mu \leq \kappa \implies 2^\mu \leq 2^\kappa$$

together with the inequalities

$$(d2) \qquad \aleph_0^\kappa = 2^\kappa \geq \operatorname{cof} 2^\kappa > \kappa$$

The behaviour of the function $\nu \mapsto 2^\nu$ on regular cardinal is completely free apart of the above constraints, namely

Theorem. *(Easton) Let $F : Reg \to Card$ be a (class) function satisfying (d1) and (d2). Then it is consistent with ZFC that $2^\nu = F(\nu)$ for all regular cardinals ν.*

1.4. **The singular cardinal case.**

Lemma. *Put $\gimel(\xi) = \xi^{\operatorname{cof}\xi}$ and $2^{<\nu} = \sup\{2^\xi \mid \xi < \nu\}$. Then*

$$(d3) \qquad 2^\nu = (2^{<\nu})^{\operatorname{cof}\nu} \text{ for all cardinals } \nu.$$

It follows

Theorem. *(Buchowski-Hechler) Let ν be singular; then*

$$(d4) \qquad 2^\nu = \begin{cases} 2^{<\nu} & \text{if } \exists\kappa < \nu\ \forall\xi\ (\kappa \le \xi < \nu \ \Rightarrow\ 2^\xi = 2^\kappa) \\ \gimel(2^{<\nu}) & \text{otherwise.} \end{cases}$$

So the power 2^ν for singular ν is determined by the function $\gimel$ on singular cardinals, together with the power $2^\kappa = \kappa^\kappa = \gimel(\kappa)$ of regular cardinals $\kappa < \nu$ (in fact $2^{<\nu} = \sup\{2^{\xi^+} \mid \xi < \nu\}$ for singular ν).

1.5. **Cardinal power.** The cardinal exponentiation κ^ν satisfies the obvious relations

$$(e1) \qquad \lambda \le \kappa \implies \lambda^\nu \le \kappa^\nu$$

$$(e2) \qquad \mu \le \nu, \implies \kappa^\mu \le \kappa^\nu$$

$$(e3) \qquad \xi < \kappa,\ \xi^\nu \ge \kappa \implies \xi^\nu = \kappa^\nu$$

together with the strict inequalities

$$(e4) \qquad \operatorname{cof}\kappa^\nu > \nu \quad \text{and} \quad \kappa^{\operatorname{cof}\kappa} > \kappa$$

It turns out that the *gimel* function $\gimel(\kappa) = \kappa^{\operatorname{cof}\kappa}$ completely determines the cardinal exponentiation.

(But clearly $\gimel(\kappa) = \kappa^\kappa = 2^\kappa$ for regular κ.)

Lemma. *Assume that $\nu < \operatorname{cof}\kappa$ and let $f : \nu \to \kappa$ be given. Then there exists $\alpha \in \kappa$ s.t. $f[\nu] \subseteq \alpha$, whence ${}^\nu\kappa \subseteq \bigcup_{\alpha\in\kappa} {}^\nu\alpha$. Hence*

$$(e5) \qquad \nu < \operatorname{cof}\kappa \implies \kappa^\nu = \sum_{\xi<\kappa} \xi^\nu \xi^+$$

It follows the *Hausdorff formula* $(\kappa^+)^\nu = \kappa^\nu \kappa^+$ and in general

$$(e6) \qquad (\aleph_{\alpha+n})^\nu = \aleph_\alpha^\nu \aleph_{\alpha+n} \quad \textit{for all } \alpha \textit{ and all } n$$

Lemma. *Let κ be a limit cardinal, and let $\nu \geq \operatorname{cof} \kappa$. Then*

$$(e7) \qquad \nu \geq \operatorname{cof} \kappa \implies \kappa^\nu = (\sup_{\xi<\kappa} \xi^\nu)^{\operatorname{cof} \kappa} \qquad (\kappa \textit{ limit})$$

Theorem. *(Buchowski)*

$$(e8) \quad \kappa^\nu = \begin{cases} 2^\nu & \textit{if } \kappa \leq 2^\nu \ \ (\textit{in particular if } \nu \geq \kappa), \\ \kappa & \textit{if } \nu < \operatorname{cof} \kappa \textit{ and } \forall \xi < \kappa \, (\xi^\nu \leq \kappa), \\ \daleth(\kappa) & \textit{if } \kappa > \nu \geq \operatorname{cof} \kappa \textit{ and } \forall \xi < \kappa \, (\xi^\nu < \kappa), \\ \daleth(\zeta) & \textit{otherwise, where } \zeta = \min \{\xi < \kappa \mid \xi^\nu \geq \kappa\}. \end{cases}$$

Remark that the last two cases may occur only when κ, resp. ζ are singular. The function $\daleth$ is not required for regular κ (actually $\daleth(\kappa) = 2^\kappa$ for regular κ.)

1.6. **Special hypotheses.** Assuming the **Generalized Continuum Hypothesis**

(GCH) $\qquad 2^\kappa = \kappa^+ \qquad$ for all infinite κ

all cardinal powers are determined, and assume the least consistent value, namely

Corollary ((GCH)).
$$\kappa^\nu = \begin{cases} \kappa & \textit{if } \ \nu < \operatorname{cof} \kappa, \\ \kappa^+ & \textit{if } \ \kappa > \nu \geq \operatorname{cof} \kappa, \\ \nu^+ & \textit{if } \nu \geq \kappa. \end{cases}$$

GCH being notoriously (almost) totally independent on regular cardinals, one formulated the **Singular Cardinals Hypothesis**

(SCH) $\qquad 2^{\operatorname{cof} \kappa} < \kappa \implies \kappa^{\operatorname{cof} \kappa} = \kappa^+ \quad$ for all singular κ

Assuming (SCH), all cardinal powers are determined, and assume the least values consistent with the powes 2^ν of the regular cardinals ν, namely

Corollary ((SCH)).

(i) for all κ, ν
$$\kappa^\nu = \begin{cases} 2^\nu & \textit{if } \ \kappa \leq 2^\nu \ \ (\textit{in part. if } \nu \geq \kappa), \\ \kappa & \textit{if } \ \nu < \operatorname{cof} \kappa \ \textit{ and } \ 2^\nu < \kappa, \\ \kappa^+ & \textit{if } \ \kappa > \nu \geq \operatorname{cof} \kappa \ \textit{ and } \ 2^\nu < \kappa. \end{cases}$$

(ii) for singular ν
$$2^\nu = \begin{cases} 2^{<\nu} & \textit{if } \ \exists \kappa < \nu \ 2^\kappa = 2^{<\nu}, \\ (2^{<\nu})^+ & \textit{otherwise.} \end{cases}$$

1.7. **Tarski's theorem on products.**

Theorem (Tarski). *Let ν be an infinite cardinal, and let the ν-sequence of cardinals $\langle \kappa_\alpha \mid \alpha < \nu \rangle$ be weakly increasing,* i.e. *s.t.* $0 < \kappa_\alpha \leq \kappa_\beta$ *for* $\alpha < \beta < \nu$. *Then*

$$\prod_{\gamma<\nu} \kappa_\gamma = (\sup_{\gamma<\nu} \kappa_\gamma)^\nu. \tag{e9}$$

Remark that the conditions of *weak monotonicity* and of *cardinal length* are always separately satisfiable, but not both together, in general.

1.8. **Shelah's pcf theory.** Let $a \subseteq Reg$ be a set of regular cardinals, which we assume to be an interval $[\aleph_\alpha, \aleph_\delta) \cap Reg$ of length $|a| < \aleph_\alpha$. Define

$pcf(a) = \{\mathrm{cof}\,(\prod_{\kappa\in a} \kappa/\mathcal{D}) \mid \mathcal{D}\ ultrafilter\ on\ a\}$, and
$pcf_\mu(a) = \bigcup\{pcf(b) \mid b \subseteq a, |b| \leq \mu\}$, for $\mu \leq |a|$

Lemma. *For all $\mu \leq |a|$:*

1. $a \subseteq pcf_\mu(a)$, *and* $\sup pcf_\mu(a) \leq (\sup a)^\mu$;
2. $\min pcf_\mu(a) = \min a$.

The following theorems are the essential part of Shelah's pcf theory (their elementary, but very complicated, proofs are contained in Holz, Steffens, and Weitz, ch 6,7, 8,9).

Let $a = [\aleph_\alpha, \aleph_\delta) \cap Reg$ and $\mu \leq |a| < \aleph_\alpha$. Then

Theorem 1.1. $pcf_\mu(a) = [\aleph_\alpha, \aleph_\gamma] \cap Reg$,
with $\aleph_\gamma$ regular $\geq \aleph_\delta$ and $|\gamma \setminus \alpha| \leq |\delta \setminus \alpha|^\mu$.

Theorem 1.2. *If $\kappa^\mu < \aleph_\alpha$ for all $\kappa < \aleph_\alpha$, then $\aleph_\gamma = \aleph_\delta^\mu$.*

Theorem 1.3. $|pcf_\mu(a)| \leq |a|^{+++} \leq |\delta|^{+++}$.

Recall that $a = [\aleph_\alpha, \aleph_\delta) \cap Reg$ and $\mu \leq |a| < \aleph_\alpha$.

Corollary. *Let δ be limit. Then*

$$\kappa^\mu < \aleph_\alpha\ for\ all\ \kappa < \aleph_\alpha \implies \aleph_\delta^\mu < \aleph_{\alpha+|pcf_\mu(a)|^+},$$

hence $\mu < \aleph_\delta \implies \aleph_\delta^\mu < \aleph_{(|\delta|^\mu)^+}$.

In particular, when $\aleph_\delta$ is a singular strong limit cardinal, then $2^{\aleph_\delta} = \beth(\aleph_\delta) < \aleph_{(2^{|\delta|})^+}$.

Corollary. *In general, for all limit ordinal δ:*

$$\beth(\aleph_\delta) \leq {\aleph_\delta}^{|\delta|} < \max\{\aleph_{|\delta|^{++++}}, (2^{|\delta|})^+\}$$

A remarkable consequence is the stunning estimate

$$2^{\aleph_0} < \aleph_\omega \implies \aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$$

2. Small large cardinals

The cardinal κ is (strongly) inaccessible if

$$\kappa \textit{ is regular}, \textit{ i.e. } \operatorname{cof} \kappa = \kappa, \quad \text{and}$$

$$\kappa \textit{ is strong limit}, \textit{ i.e. } \mu < \kappa \Longrightarrow 2^\mu < \kappa.$$

Hence κ cannot be a successor, so it is a regular limit cardinal (the latter are now called weakly inaccessible cardinals)

According to the definition, ω is an inaccessible cardinal. On the other hand, if $\kappa > \omega$, then the corresponding segment V_κ of the cumulative hierarchy is a transitive model of ZFC, so the existence, and even the consistency, of uncountable inaccessible cardinals cannot be proved in ZFC.

2.1. **trees.** A tree $(T, <)$ is a partially ordered set s.t. the predecessors of any $t \in T$ are well ordered by $<$. The αth level T_α of T is the set of all $t \in T$ s.t. the order type of the predecessors of t is α. The height $h(T)$ of T is the least α s.t. $T_\alpha = \emptyset$. A subset of T totally ordered by $<$ in order-type (length) α is an α-path. A branch of T is a $h(T)$-path, *i.e.* one of maximal length. A κ-tree is tree of height κ whose levels have size less than κ. A cardinal κ has the tree property if every κ-tree has a κ-branch. A classical "infinitary" property of ω is the tree property.

Theorem. *(König's Lemma) Any infinite tree whose levels are all finite has an infinite branch.*

2.2. **Partition relations.** Denote by $[X]^n = \{Y \subseteq X \mid |Y| = n\}$, *i.e.* the set of all (unordered) n-tuples of elements of X.

The partition relation $\kappa \to (\lambda)^n_s$ means that any partition (coloring) of $[\kappa]^n$ into s parts (colors) admits a homogeneous set, *i.e.* a subset $H \subseteq \kappa$ s.t. $[H]^n$ is monochromatic (all n-tuples from H belong to the same part of the partition). Another classical "infinitary" property of ω is the partition property.

Theorem. *(Ramsey)* $\omega \to (\omega)^n_s$ *for all* $n, s < \omega$.

Clearly
$\nu \geq \kappa,\ \mu \leq \lambda,\ m \leq n,\ t \leq s \implies (\kappa \to (\lambda)^n_s \implies \nu \to (\mu)^m_t)$
On the other hand $\kappa \not\to (\omega)^2_\kappa$, and $\kappa \not\to (\omega)^\omega_2$.
Moreover $2^\kappa \not\to (\omega)^2_\kappa$, and $2^\kappa \not\to (\kappa^+)^2_2$.
Hence $\kappa \to (\kappa)^2_2 \implies \kappa$ *strongly inaccessible.*

2.3. **Weakly compact cardinals and languages.** Call weakly compact a cardinal κ s.t. $\kappa \to (\kappa)^2_2$. Weakly compact cardinals refer to a property of the infinitary languages $\mathcal{L}_{\kappa,\lambda}$ with κ variables, where conjunctions and disjunctions of length less than κ, and universal and existential quantifications on blocks of less than λ variables are permitted.

The language $\mathcal{L}$ is strongly κ-compact when any set Σ of sentences of $\mathcal{L}$ has a model if and only if any subset of Σ of size less than κ has a model. The language $\mathcal{L}$ is weakly κ-compact if any set Σ of sentences of $\mathcal{L}$ of size $\leq \kappa$ has a model if and only if any subset of Σ of size less then κ has a model.

Ramsey's theorem implies the compactness theorem of first-order classical logic:

Theorem. *$\mathcal{L}_{\omega,\omega}$ is strongly ω-compact.*

2.4. **Weak compactness v/s tree property.** The following implications are straightforward:

(1) If $\mathcal{L}_{\kappa\omega}$ is weakly compact, then κ is weakly inaccessible.
(2) If κ has the tree property, then κ is regular.
(3) If $\kappa = \lambda^+$ has the tree property, then $\lambda^{<\lambda} \geq \kappa$.

Theorem.

(1) *If $\kappa \to (\kappa)^2_2$ then κ has the tree property.*
(2) *If κ is inaccessible and has the tree property, then $\kappa \to (\kappa)^n_\lambda$ for all $n \in \omega$ and for all $\lambda < \kappa$.*
(3) *If κ is inaccessible and $\mathcal{L}_{\kappa,\omega}$ is weakly κ-compact, then κ has the tree property .*
(4) *If κ is inaccessible and has the tree property, then $\mathcal{L}_{\kappa,\kappa}$ is weakly compact.*

Hence, assuming GCH, $\mathcal{L}_{\kappa\omega}$ is weakly compact if and only if κ is weakly compact, and only successors of singular cardinals might have the tree property without being weakly compact. (and this would require very large cardinals, implying $V \neq L$).

A sufficient condition for obtaining the equivalence between tree property and weak compactness, without assuming GCH, is the combinatorial principle $\square_\kappa$ for all κ.

2.5. **Partition properties with ordinal goals.** Consider the finer partition relation $\kappa \to (\alpha)^n_\lambda$, with α a (non-necessarily initial) ordinal, meaning that for all $\varphi : [\kappa]^n \to \lambda$ there is a homogeneous H of order type α.

Lemma. *(Stepping up lemma)*
For $1 \le n < \omega$ *and* $\lambda \le 2^{<\kappa} = (2^{<\kappa})^{<\kappa}$ *(in particular* $\lambda < \operatorname{cof}\kappa$*)*

$$\kappa \to (\alpha)^n_\lambda \implies (2^{<\kappa})^+ \to (\alpha+1)^{n+1}_\lambda \quad .$$

Recall that the $\beth$-hierarchy is defined inductively as
$\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$, $\beth_\lambda(\kappa) = \sup_{\alpha<\lambda} \beth_\alpha(\kappa)$ (*limit* λ)

Theorem. *(Erdös-Rado partition theorem)*
$\beth_n(2^{<\kappa})^+ \to (\kappa+n+1)^n_\lambda$ *for all* $\lambda < \operatorname{cof}\kappa$.
Hence $\beth_n(\kappa)^+ \to (\kappa^+ + n)^n_\kappa$ *for all* κ.

So, in particular, $2^\kappa \not\to (\kappa^+)^2_2$, but $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.
Partition relations with infinite exponent being impossible, consider the partition relation $\kappa \to (\alpha)^{<\omega}_\lambda$, meaning that for any λ-colouring $\varphi : [\kappa]^{<\omega} \to \lambda$ of all finite parts of κ, there exists a set $H \subseteq \kappa$ of order type α homogeneous for φ, *i.e.* such that each set $[H]^n$ is (separately) monochromatic for $\varphi_{|[\kappa]^n}$, $2 \le n < \omega$.
The αth Erdös cardinal $\kappa(\alpha)$ is the (necessarily uncountable) cardinal $\kappa(\alpha) = \min\{\kappa \mid \kappa \to (\alpha)^{<\omega}_2\}$.
Caveat $\kappa \to (\alpha)^{<\omega}_\lambda \implies \forall n < \omega\, (\kappa \to (\alpha)^n_\lambda)$, but the implication cannot be reversed, *e.g.* $\omega \not\to (\omega)^{<\omega}_2$.

Theorem.

(1) $\kappa(\alpha)$ *is regular, and* $\kappa(\alpha) \not\to (\alpha+1)^{<\omega}_2$;
(2) *for limit* α, $\kappa(\alpha)$ *is inaccessible and* $\forall\lambda<\kappa(\alpha)\,(\kappa(\alpha) \to (\alpha)^{<\omega}_\lambda)$;
(3) $\kappa(\alpha+n+1) = \beth_n(\kappa(\alpha))^+$ *for all* $n < \omega$.

Let M be a model for the language $\mathcal{L}$. A set $\mathcal{I} \subseteq \kappa$ is a set of indiscernibles for the model $M \supseteq \kappa$ of $\mathcal{L}$ if for any formula ϕ of $\mathcal{L}$ with $x_1, \ldots, x_n$ free, and any increasing sequences $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ from I:
$M \models \phi[\alpha_1, \ldots, \alpha_n] \iff M \models \phi[\beta_1, \ldots, \beta_n]$

Remark. Let $\varphi : [\kappa]^{<\omega} \to \{0,1\}$, and put $\varphi_n = \varphi_{|[\kappa]^n}$. Then any set of indiscernibiles for the model $M = (V_\kappa; <, \{\varphi_n \mid n < \omega\})$ is homogeneous for φ.

Lemma. *If* $\kappa \to (\alpha)^{<\omega}_{2^{|\mathcal{L}|}}$ *there exists a set of indiscernibles for* $\mathcal{L}$ *of order type* α.

Corollary. $\kappa(\alpha) \to (\alpha)^{<\omega}_\lambda$ *for all* $\lambda < \kappa(\alpha)$.

The strongest partition property leads to call a cardinal κ Ramsey if $\kappa \to (\kappa)^{<\omega}_2$ or equivalently if $\kappa \to (\kappa)^{<\omega}_{<\kappa}$

3. Large large cardinals

3.1. **Ideals, filters, and measures.** A nonempty family of subsets $\mathcal{I} \subseteq \mathcal{P}(I)$ is an ideal on I if

(1) $B \in \mathcal{I},\ \ A \subseteq B \implies \mathcal{A} \in \mathcal{I}$;
(2) $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$;
(3) $I \notin \mathcal{I}$.

The ideal $\mathcal{I}$ is κ-complete if it is closed under unions of size less than κ, *i.e.* $\{A_\alpha \mid \alpha < \lambda < \kappa\} \subseteq \mathcal{I} \implies \bigcup_{\alpha<\lambda} A_\alpha \in \mathcal{I}$.

The ideal $\mathcal{I}$ is a κ-saturated if every disjoint family of sets not in $\mathcal{I}$ has size less then κ, *i.e.* $\mathcal{A} \cap \mathcal{I} = \emptyset$, $\forall A, B \in \mathcal{A}\,(A \cap B = \emptyset) \Longrightarrow |\mathcal{A}| < \kappa$.

The ideal $\mathcal{I}$ is prime if it is maximal, or equivalently $A \in \mathcal{I} \Leftrightarrow I \setminus A \notin \mathcal{I}$.

A nonempty family of subsets $\mathcal{F} \subseteq \mathcal{P}(I)$ is a filter on I if

(1) $A \in \mathcal{F},\ \ A \subseteq B \implies B \in \mathcal{F}$;
(2) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$;
(3) $\emptyset \notin \mathcal{F}$.

The filter $\mathcal{F}$ is κ-complete if it is closed under intersections of size $\lambda < \kappa$, *i.e.* $\{A_\alpha \mid \alpha < \lambda < \kappa\} \subseteq \mathcal{F} \implies \bigcap_{\alpha<\lambda} A_\alpha \in \mathcal{F}$.

The filter $\mathcal{F}$ is a principal if there exists $E \subseteq I$ s.t. $A \in \mathcal{F} \Leftrightarrow E \subseteq A$.

A filter $\mathcal{U}$ is an ultrafilter if it is maximal, equivalently $A \in \mathcal{U} \Leftrightarrow I \setminus A \notin \mathcal{U}$. A principal filter $\mathcal{F}_E$ is ultra if and only if $E = \{i\}$ is a singleton.

$\mu : \mathcal{P}(X) \to \mathbb{R}$ is a (nontrivial, σ-additive) measure on X if

(1) $\mu(\{x\}) = 0$ for all $x \in X$;
(2) $B \subseteq A \implies \mu(B) \leq \mu(A)$;
(3) $A_n \cap A_m = \emptyset$ for all $m \neq n \implies \mu(\bigcup_{n<\omega} A_n) = \sum_{n<\omega} \mu(A_n)$.

μ is κ-additive if $\mu(A_\alpha) = 0$ for $\alpha < \lambda < \kappa \Longrightarrow \mu(\bigcup_{\alpha<\lambda} A_\alpha) = 0$.

μ is two-valued if $\mu : \mathcal{P}(X) \to \{0, 1\}$.

An atom of μ is $A \subseteq X$ s.t. $B \subseteq A$, $\mu(B) \neq \mu(A) \Longrightarrow \mu(B) = 0$.

μ is atomless if there are no atoms.

Theorem. *(Ulam 1930ca.) If μ is an atomless measure on X, then X is the union of $\leq 2^{\aleph_0}$ zero-sets, and $2^{\aleph_0} \geq$ the least weakly inaccessible cardinal. If μ has an atom, then $|X| \geq$ the least (strongly) inaccessible cardinal.*

For $n < \omega$ any disjoint family of sets of measure $\geq \frac{1}{n}$ has size $\leq n$ (if μ is two-valued). Hence, if μ is κ-additive and $\{A_\alpha \mid \alpha < \lambda < \kappa\}$ is a family of pairwise disjoint sets, then $\mu(\bigcup_{\alpha<\lambda} A_\alpha) = \sup_{x \in [\lambda]^{<\omega}} \sum_{\alpha \in x} \mu(A_\alpha)$, so the measure is properly "κ-additive".

The zero-ideal of μ is $\mathcal{I}_\mu = \{A \subseteq X \mid \mu(A) = 0\}$, which is is κ-complete iff μ is κ-additive, and is prime iff μ is two-valued.

Similarly, the set $\mathcal{F}_\mu = \{A \subseteq X \mid \mu(A) = 1\}$ is a filter, which is is κ-complete iff μ is κ-additive, and is an ultrafilter iff μ is two-valued.

The family of all non-zero sets $\mathcal{I}_\mu^+ = \{A \subseteq X \mid \mu(A) > 0\}$ (the complement of $\mathcal{I}_\mu$) is a filter (actually an ultrafilter) iff $\mathcal{I}_\mu^+ = \mathcal{F}_\mu$.

Lemma. *Let κ be the least cardinal carrying a measure μ: then μ is κ-additive if and only if the corresponding ideal $\mathcal{I}_\mu$ is κ-complete. Moreover, μ is two valued if and only if $\mathcal{I}_\mu^+$ is a κ-complete ultrafilter $\mathcal{U}_\mu$.*

Lemma. *Let μ be an atomless measure on κ: then there is a partition of κ in no more than $2^{\aleph_0}$ null sets. Hence μ induces a measure on some $\lambda \leq 2^{\aleph_0}$, and so also on $\mathbb{R}$.*

3.2. **Measurable cardinals.** Call κ measurable if there is a two valued κ-additive measure on κ. Call κ real-valued measurable if there is any κ-additive measure on κ.

Theorem. *If κ is measurable, then κ is inaccessible. If κ is real-valued measurable, then κ is weakly inaccessible.*

Let $\mathcal{U}$ be an ultrafilter on I and let $\{M_i \mid i \in I\}$ be indexed by I. Then the ultraproduct $\prod_{i\in I} M_i/\mathcal{U}$ is the quotient of the Cartesian product $\prod_{i\in I} M_i$ modulo the equivalence $\equiv_\mathcal{U}$ defined by

$$\langle x_i \mid i \in I\rangle \equiv_\mathcal{U} \langle y_i \mid i \in I\rangle \iff \{i \in I \mid x_i = y_i\} \in \mathcal{U}.$$

Similarly, membership mod $\mathcal{U}$ is defined on $\prod_{i\in I} M_i/\mathcal{U}$ by

$$\langle x_i \mid i \in I\rangle \in_\mathcal{U} \langle y_i \mid i \in I\rangle \iff \{i \in I \mid x_i \in y_i\} \in \mathcal{U}.$$

Then, for any set-theoretic formula ϕ, one has

Theorem. *(Los) For all $f_1, \ldots, f_n \in \prod_{i\in I} M_i$*

$$\prod_{i\in I} M_i/\mathcal{U} \models \phi[f_1, \ldots, f_n] \iff \{i \in I \mid M_i \models \phi[f_1(i), \ldots, f_n(i)]\} \in \mathcal{U}\ .$$

When all sets M_i are equal to the same set M, the ultraproduct $\prod_{i\in I} M_i/\mathcal{U}$ is called the ultrapower of M modulo $\mathcal{U}$ and denoted by $M^I_\mathcal{U}$. The diagonal embedding $d_\mathcal{U} : M \to M^I_\mathcal{U}$ maps any $x \in M$ to the equivalence class of the constant function $c_x : i \mapsto x$.

Then Los' theorem implies, for any set-theoretic formula ϕ,

Corollary. *For all $x_1, \ldots, x_n \in M$*

$$M^I_\mathcal{U} \models \phi[d_\mathcal{U}(x_1), \ldots, d_\mathcal{U}(x_n)] \iff M \models \phi[x_1, \ldots, x_n]\ .$$

When M is a proper class, the equivalence classes may be proper, hence uncollectible, but one can use Scott's trick and extract from each class the elements of least rank, which are a set characterizing the whole equivalence class. Then the ultrapower $M^I_\mathcal{U}$ remains a proper class, but its elements are sets.

3.3. **Mostowski's collapse.** If the relation $\in_{\mathcal{U}}$ is extensional and wellfounded, one has the Mostowski collapse, *i.e.* the unique isomorphism $\pi : M^I_{\mathcal{U}} \to N$ onto a transitive class.

Lemma. *When M is transitive, the relation $\in_{\mathcal{U}}$ is extensional, and it is wellfounded iff the ultrafilter $\mathcal{U}$ is $\aleph_1$-complete.*

Then the composition $j_{\mathcal{U}} = \pi \circ d_{\mathcal{U}} : M \to N$ is an elementary embedding s.t. $M \models \phi[x_1, \ldots, x_n] \iff N \models \phi[j_{\mathcal{U}}(x_1), \ldots, j_{\mathcal{U}}(x_n)]$. In particular $\forall x, y \in M \ (j_{\mathcal{U}}(x) \subseteq j_{\mathcal{U}}(y) \iff x \subseteq y)$.

If κ is measurable, the ultrafilter $\mathcal{U}_\mu$ is κ-complete, so there exists a transitive class M and a unique nontrivial elementary embedding $\pi \circ d_{\mathcal{U}_\mu} = j_\mu : V \to M$.

Any $f : \kappa \to \kappa$ not $\mathcal{U}_\mu$-equivalent to a constant c_α, with $\alpha < \kappa$, is mapped by π to an ordinal $\pi(f) < \pi(c_\kappa) = j_\mu(\kappa)$. On the other hand, by induction on α, $j_\mu(\alpha) = \pi(c_\alpha) = \alpha$ for all $\alpha < \kappa$. Hence $\kappa < j_\mu(\kappa)$. (Actually $j_\mu(x) = x$ for all $x \in V_\kappa$, and so $j_\mu(V_\kappa) = V_\kappa$.)

More generally, any nontrivial elementary embedding $j : V \to M$ onto a transitive class M has a critical point $\kappa = \operatorname{crit} j$, the least ordinal moved by j, s.t. $j_\mu(\kappa) > \kappa$, while $j_\mu(\alpha) = \alpha$ for all $\alpha < \kappa$.

3.4. **Normal ultrafilters.** A nonprincipal κ-complete ultrafilter $\mathcal{U}$ on κ is normal if it is closed under diagonal intersections, *i.e.*
$\forall \alpha < \kappa. U_\alpha \in \mathcal{U} \implies \Delta_{\alpha<\kappa} U_\alpha = \{\beta < \kappa \mid \beta \in \bigcap_{\alpha<\beta} U_\beta\} \in \mathcal{U}$, or eqivalently any regressive function $f \in \kappa^\kappa$ is almost constant mod $\mathcal{U}$, *i.e.* $(\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \mathcal{U} \implies \exists \beta. \{\alpha < \kappa \mid f(\alpha) = \beta\} \in \mathcal{U})$.

Lemma.
Let $[h]$ be the least nonconstant ordinal function in $V^\kappa/\mathcal{U}$ (so that $\pi[h] = \kappa$)). Then both conditions
(i) $[id_\kappa] = [h]$, and (ii) $\forall U \subseteq \kappa \ (U \in \mathcal{U} \iff \kappa \in j_{\mathcal{U}}(U))$
are equivalent to normality.

Theorem. *Let $j : V \to M$ be an elementary embedding with $crit(j) = \kappa$. Then the set $\mathcal{U} = \{U \subseteq \kappa \mid \kappa \in j(U)\}$ is a normal κ-complete ultrafilter on κ, and $[id_\kappa]_{\mathcal{U}}$ is the least class of non-constant functions in $\kappa^\kappa/\mathcal{U}$. Let N be the Mostowski collapse of the ultrapower $V^\kappa_{\mathcal{U}}$: then the map $k : N \to M$ such that $k : \pi([f]_{\mathcal{U}}) \mapsto (j(f))(\kappa)$ is an elementary embedding that makes the following diagram commute*

$$\begin{tikzcd}
 & V \arrow[dl, "d"'] \arrow[d, "d_{\mathcal{U}}"] \arrow[r, "j"] \arrow[dr, "j_{\mathcal{U}}"] & M \\
V^\kappa \arrow[r, "{[\ ]_{\mathcal{U}}}"] & V^\kappa_{\mathcal{U}} \arrow[r, "\pi", "\cong"'] & N \arrow[u, "k"']
\end{tikzcd}$$

(where d is the "diagonal" map and $[\ ]_{\mathcal{U}}$ is the projection onto the quotient mod $\mathcal{U}$*)*

Moreover one has

(1) $j_{\mathcal{U}}(x) = x$ *for all* $x \in V_\kappa$*, and* $j_{\mathcal{U}}(X) \cap V_\kappa = X$ *for all* $X \subseteq V_\kappa$*;*
(2) $\mathcal{U} \notin N$*, and* $N^\kappa \subseteq N$*, but* $N^{\kappa^+} \not\subseteq N$*;*
(3) $2^\kappa \leq (2^\kappa)^N < j_{\mathcal{U}}(\kappa) < (2^\kappa)^+$*;*
(4) *for* λ *limit ordinal,* $\operatorname{cof} \lambda = \kappa \Longrightarrow j_{\mathcal{U}}(\lambda) > \bigcup_{\alpha<\lambda} j_{\mathcal{U}}(\alpha)$*,*
and $\operatorname{cof} \lambda \neq \kappa \implies j_{\mathcal{U}}(\lambda) = \bigcup_{\alpha<\lambda} j_{\mathcal{U}}(\alpha)$*.*

Corollary. *Let μ be a normal measure on κ: then any partition of $[\kappa]^{<\omega}$ into less than κ parts has a homogeneous set of measure* 1*, hence every measurable cardinal is Ramsey. Actually, almost all ordinals less than κ are Ramsey (and* a fortiori *weakly compact).*

3.5. **Compact cardinals.** Let κ be regular and $\lambda \geq \kappa$. A κ-complete filter $\mathcal{F}$ on $[\lambda]^{<\kappa}$ is fine if, for all $\alpha < \lambda$, the cone $C(\alpha) = \{x \in [\lambda]^{<\kappa} \mid \alpha \in x\} \in \mathcal{F}$.
A fine ultrafilter $\mathcal{U}$ on $[\lambda]^{<\kappa}$ is is normal if any choice function $f : [\lambda]^{<\kappa} \to \lambda$ is constant on some $U \in \mathcal{U}$. or equivalently $\mathcal{U}$ is closed under diagonal intersections, *i.e.*

$$\forall\alpha < \kappa. U_\alpha \in \mathcal{U} \implies \Delta_{\alpha<\kappa} U_\alpha = \{x \in [\lambda]^{<\kappa} \mid x \in \textstyle\bigcap_{\alpha\in x} U_\alpha\} \in \mathcal{U}.$$

A cardinal κ is λ-compact if there is a fine ultrafilter on $[\lambda]^{<\kappa}$, and κ is λ-supercompact if there is a normal ultrafilter on $[\lambda]^{<\kappa}$; then κ is [super]compact if it is λ-[super]compact for all $\lambda \geq \kappa$.

Any measurable cardinal κ is κ-supercompact. (if $\mathcal{U}$ is a normal ultrafilter on κ, then $\{X \subseteq [\kappa]^{<\kappa} \mid X \cap \kappa \in \mathcal{U}\}$ is normal)

CAVEAT: ω is compact, but not even ω-supercompact.

Clearly any compact cardinal is measurable, and any supercompact cardinal is compact, but the reverse implications are neither provable nor refutable. Actually, there is a model where there is exactly one measurable cardinal, which is also compact, and there is another one where there is exactly one compact cardinal, which is also supercompact.

Theorem. *The following properties are equivalent for regular κ:*

(1) *every κ-complete filter on any set X of size $\geq \kappa$ is contained in some κ-complete ultrafilter on X;*
(2) *κ is (strongly) compact;*
(3) *the compactness theorem holds for the language $\mathcal{L}_{\kappa\omega}$ (or equivalently for $\mathcal{L}_{\kappa\kappa}$).*

Call (κ,λ)-regular a κ-complete nonprincipal ultrafilter $\mathcal{U}$ on λ if there is a family $\{X_\alpha \in [\lambda]^{<\kappa} \mid \alpha < \lambda\}$ s.t., for all $\beta < \lambda$, $\{\alpha < \lambda \mid \beta \in X_\alpha\} \in \mathcal{U}$.

Lemma. *If $\lambda > \kappa$ are regular and there is a fine ultrafilter on $I = [\lambda]^{<\kappa}$, then there is a (κ,λ)-regular ultrafilter on λ.*

Theorem. *(Solovay) The equality $\lambda^{<\kappa} = \lambda$ holds for all regular λ above the least compact cardinal κ. It follows that the singular cardinal hypothesis SCH holds above the least compact cardinal.*

3.6. **λ-supercompact and η-extendible cardinals.**

Lemma. *Let $\lambda \geq \kappa$ be regular, let $\mathcal{U}$ be a normal ultrafilter on $I = [\lambda]^{<\kappa}$, and let $j_{\mathcal{U}} = j : V \to M$ be the elementary embedding onto the Mostowski collapse of the ultrapower $V^I_{\mathcal{U}}$.*

Then $G = \pi[id_I] = \{j(\alpha) \mid \alpha < \lambda\}$ and $\mathcal{U} = \{U \subseteq I \mid G \in j(U)\}$. Moreover $crit(j) = \kappa = j(i \mapsto i \cap \kappa) < \lambda = j(i \mapsto o.\text{-}t.\, i)$, and $M^\lambda \subseteq M$.

Theorem. *Let $j : V \to M$ be an elementary embedding with $crit(j) = \kappa$. Then there is $\lambda \geq \kappa$ s.t. $M^\lambda \subseteq N$ if and only if κ is λ-supercompact.*

Corollary. *Let κ be 2^κ-supercompact. Then κ is the κth measurable cardinal. Actually there is a normal measure on κ s.t. almost all ordinals less than κ are measurable.*

A cardinal κ is η-extendible if $\exists \beta\, \exists j : V_{\kappa+\eta} \to V_\beta$ with $crit(j) = \kappa$, $\eta < j(\kappa)$, and κ is extendible if it is η-extendible for all η, or equivalently $\forall \alpha > \kappa\, \exists \beta\, \exists j : V_\alpha \to V_\beta$ with $crit(j) = \kappa$.

Clearly κ η-extendible $\Longrightarrow \kappa$ δ-extendible for all $\delta < \eta$.

Lemma. *Assume κ λ-supercompact, and $\nu < \kappa$ δ-supercompact for all $\delta < \kappa$: then ν is λ-supercompact.*

Theorem.

1. *If κ is $\beth(\kappa+\eta)$-supercompact and $\eta < \kappa$, then almost all $\alpha < \kappa$ are η-extendible.*
2. *If κ is η-extendible and $\eta \geq \lambda + 2$, then κ is $\beth(\kappa+\lambda)$-supercompact.*
3. *If κ is 1-extendible and supercompact, then almost all $\alpha < \kappa$ are supercompact.*
4. *If κ is extendible, then almost all $\alpha < \kappa$ are supercompact.*

3.7. **Largest (not proved inconsistent) cardinals.** Let j elementary, $\kappa = crit(j)$, $j^{n+1}(\kappa) = j(j^n(\kappa)$, $j^\omega(\kappa) = \sup j^n(\kappa)$.

1. κ is superhuge if $\forall \eta \exists j : V \to M$ with $\eta < j(\kappa)$, $M^{j(\kappa)} \subseteq M$;
2. κ is n-huge if $\exists j : V \to M$ with $M^{j^n(\kappa)} \subseteq M$;

(3) κ is ω-huge if κ is n-huge for all $n < \omega$.
(4) κ is $I1$ or $I3$ if $\exists\lambda\,\exists j : V_\lambda \to V_\lambda$ (and then necessarily either $\lambda = j^\omega(\kappa) + 1$ or $\lambda = j^\omega(\kappa)$, resp.).

Theorem.

(1)
- *κ is 1-huge $\iff$ there is a normal ultrafilter on $[\kappa]^\omega$.*
- *κ least (1-)huge cardinal $\Rightarrow$ $\kappa <$ least supercompact;*
- *κ-superhuge $\Rightarrow$ almost all $\alpha < \kappa$ are extendible;*
- *κ 2-huge $\Rightarrow$ almost all $\alpha < \kappa$ are superhuge;*
- *κ $(n+1)$-huge $\Rightarrow$ almost all $\alpha < \kappa$ are n-huge.*

(2) *If κ is $I1$, then almost all $\alpha < \kappa$ are $I3$, and if κ is $I3$, then almost all $\alpha < \kappa$ are ω-huge.*

3.8. **Reinhardt's cardinals.** The ultimate closure property of a large cardinal should be the existence of $j : V \to V$ with $\kappa = crit(j)$: call such a κ Reinhardt.

More demanding, call κ Berkeley if for all transitive M with $\kappa \in M$ there is $j : M \to M$ with $crit(j) < \kappa$.

Every Reinhardt cardinal is Berkeley, and Berkeley cardinals are above ω-huge cardinals. It is an open problem whether Reinhardt's cardinals are relatively consistent with $\mathsf{ZF}+\mathsf{DC}$.
However their existence contradicts the axiom of choice.

Lemma. *(Erdös-Hajnal) If $2^\kappa = \kappa^{\aleph_0}$, then there is $f : [\kappa]^\omega \to \kappa$ s.t., for any $X \in [\kappa]^\kappa$, $\kappa = \{f(x) \mid x \in [X]^\omega\}$.*

Theorem. *(Kunen) Let $j : V \to M$ be an elementary embedding with $crit(j) = \kappa$, and let $\lambda = \sup_{n<\omega} j^n(\kappa)$.*

Then $G = \{j(\alpha) \mid \alpha < \lambda\} \notin M$, hence $M^\lambda \not\subseteq M$.

Corollary. *There is no nontrivial elementary $j : V_{\lambda+2} \to V_{\lambda+2}$.*

4. The axiom of determinacy AD

In the game G_A, for $A \subseteq \omega^\omega$, two players play alternatively natural numbes: I wins if the resulting sequence belongs to A, otherwise II wins. A strategy for I is a "rule" for choosing moves $\sigma : \bigcup_{n<\omega} \omega^{2n} \to \omega$; similarly, a strategy for II is $\tau : \bigcup_{n<\omega} \omega^{2n+1} \to \omega$. G_A is determined if one player has a winning strategy. The axiom AD^P states that for each $A \in P$ one player has a winning strategy. AD is simply $\mathsf{AD}^{\mathcal{P}(\omega^\omega)}$.

Theorem.

(1) *$\mathsf{AC} \Longrightarrow \exists A \subseteq \omega^\omega$ s.t. G_A is not determined.*
(2) *$\mathsf{AD} \Longrightarrow [\mathbb{R}]^\omega$ has a choice function.*
(3) *$\mathsf{AD} \Longrightarrow$ every set of reals is Lebesgue measurable, has the Baire property, and, if uncountable, has a perfect subset.*

4.1. **Large cardinals and partial determinacy.** Let $\mathcal{B}, \mathcal{A}$, and $\mathcal{P}$ be the σ -algebras of the Borel, analytic, and projective sets, resp.

Theorem.

(1) $\mathsf{AD}^{\mathcal{B}}$ *holds in* ZFC.
(2) $\exists\kappa$ *measurable* $\Longrightarrow AD^{\mathcal{A}}$.
(3) $\exists\kappa$ *supercompact* $\Longrightarrow \mathsf{AD}^{L(\mathbb{R})} \Longrightarrow \mathsf{AD}^{\mathcal{P}}$.

((2) is Thm.105 of Jech, for (3) ω Woodin cardinals with a measurable above suffice; complete proof in Kanamori VI.32)

4.2. **Large cardinals under AD.** Since there is a set of reals of size $\aleph_1$ without perfect subsets, one has $\mathsf{AD} \implies \aleph_1 \not\leq 2^{\aleph}_0$.

Considering instead the surjective ordering of cardinalities, let
$\Theta = \sup\{\alpha \mid \exists f : \mathbb{R} \to \alpha \text{ onto}\}$ (so $\Theta = (2^{\aleph}_0)^+$ in ZFC).

Theorem (For complete proofs see Kanamori Ch. VI. 28.)**.**

(1) $\mathsf{AD} \Longrightarrow \Theta = \aleph_\Theta$ *and* $\mathsf{AD}+\mathsf{DC} \Longrightarrow \operatorname{cof}\Theta > \omega$
(2) $\mathsf{AD} \Longrightarrow \aleph_1, \aleph_2, \aleph_{\omega+1}, \aleph_{\omega+2}$ *are measurable,* $\forall n \operatorname{cof}\aleph_n = \aleph_2$.
(3) $\mathsf{AD} + V = L(\mathbb{R}) \Longrightarrow \mathsf{DC}+\operatorname{cof}\Theta = \Theta = \sup\{\alpha \,|\, \alpha \textit{ measurable}\}$

CAVEAT DC holds in $L(\mathbb{R})$, hence DC is relatively consistent with AD, but it is not implied by AD.

4.3. **Bibliography.**

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