Real Analysis
(Analisi Reale)

a.y. 2023/2024
second semester

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An introduction to the course

Real Analysis is a vast branch of Analysis which serves as the foundation for several areas of Mathematics. These include Harmonic Analysis, Fourier Analysis, Partial Differential Equations, Functional Analysis, Complex Analysis, Probability, Geometric Measure Theory, and more. We can loosely summarize Real Analysis by its main topics, as topological and metric spaces, measures, integration, differentiation, function spaces, approximation of functions and their fine properties. We mention some classical topics of basic Real Analysis which will be covered in the course: properties of measures and of the abstract Lebesgue integral, Radon–Nikodym theorem, relationship between the Riemann integral and the Lebesgue integral, correspondence between measures and outer measures, Fubini–Tonelli theorem, functions of bounded variation and absolutely continuous functions in one variable, and differentiation of monotone functions. Among the more advanced topics, we will study Banach space valued measures, the Bochner integral, Hausdorff measure, Hausdorff dimension, fractals and the area formula for the Euclidean Hausdorff measure.

The basic Real Analysis of the course clearly overlaps with the basics of Analysis and Probability, while the more advanced topics are not typically addressed in other courses. One feature of the present course is to provide detailed proofs in both simple and advanced topics, especially taking into account bachelor students. In fact, the main point of the course is to organize all the material in a self-contained manner as much as possible. Following this approach, the course provides a clear and logical structure for learners to follow. It allows them to understand the concepts and topics in a systematic way, building upon previously covered material. A self-contained organization also enables students to comprehend the material independently. They can refer back to previous topics within the course, without relying heavily on external reference sources. This promotes self-study and empowers students to grasp the content at their own pace.

About the final

The final consists of an oral examination, where the questions cover the entire program and require statements of theorems and their detailed proofs. A list of exercises will be presented and updated during the course. Some of these exercises might be proposed during the oral examination. Regarding the final evaluation, the ability to solve one of these exercises is taken into account only on the positive side, especially for nontrivial exercises.

Detailed program of the course

1. **Measures and outer measures.** Measurable spaces, outer measures and their basic properties, Carathéodory's theorem for outer measures, measurable functions and their approximation, Lebesgue measure, nonmeasurable sets, algebras, rings and semirings of sets, Carathéodory-Hahn's extension.
2. **Integration theory and Lebesgue spaces.** Lebesgue integral over a measure space, Beppo Levi's, Fatou's and Lebesgue's theorems. Continuity and differentiation of integrals with respect to parameters. Basic properties of spaces with respect to a measure µ and their completeness. Convergence in , pointwise a.e. , in measure and related implications. Jensen inequality.
3. **Operations on measures.** Signed measures, Hahn's theorem and Jordan decomposition theorem for signed measures, total variation, absolute continuity of the integral and Radon-Nikodym's theorem.
4. **Measures and topology.** Borel measures, approximation of Borel sets with open, closed and compacts sets, Carathéodory's criterion for Borel measures, Radon measures on topological spaces, Lusin theorem and density of continuous functions in functions. Riesz's representation theorem for linear functionals on the space of continuous functions with compact support.
5. **Fubini's and Tonelli's theorems.** Product of measures, Fubini's and Tonelli's theorems over measure spaces and their counterexamples with respect to weakened assumptions.
6. **Absolutely continuous functions and functions of bounded variation.** Vitali's covering theorems, a.e. differentiability of monotone functions, function of bounded variation and Cantor's function. Absolutely continuous functions and their characterization by the fundamental theorem of Calculus.
7. **Hausdorff measure, fractals and area formula.** Hausdorff measure and Hausdorff dimension, some computations of Hausdorff dimension for fractals, relationship between holder continuous functions and Hausdorff measure, equality between Hausdorff measure and Lebesgue measure, area formula and change of variable in Euclidean space.
8. **Optional material I.** Vector measures, Bochner integral and counterexamples to Radon-Nikodym theorem for Banach space valued measures. Relationship between measures and outer measures.
9. **Optional material II.** Compact-open topology, Stone's approximation theorem and separability of $C_c(X)$, where X is a separable metric space.
10. **Optional material III.** Another approach to Lebesgue integral through upper and lower sums. Comparison between Lebesgue integral and Riemann integral. Characterization of Riemann integrable functions.

Note: All topics of the syllabus will be presented along with their proofs.

Practical information

The course will last 48 hours and will take place in the second semester. Additional information is available at the webpages:

`https://esami.unipi.it/programma.php?c=58015&aa=2023&docente=MAGNANI&insegnamento=&sd=0`

`http://people.dm.unipi.it/magnani/Corsi/AnalisiReale/AnalisiReale.html`

All the students that have questions about the course are invited to contact me directly by the email address: valentino.magnani@unipi.it