Equazioni differenziali stocastiche ed applicazioni

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A stochastic differential equation (SDE) is a differential equation in which a random noise term appears. The prototype of a stochastic differential equation is given by

$$\mathrm{d}X = b(X)\mathrm{d}t + \sigma(X)\mathrm{d}W$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^{d\times m} \to \mathbb{R}^d$ are the drift and diffusion coefficient, respectively. Here, W is an m-dimensional Brownian motion, and X is the solution, a stochastic process taking values in $\mathbb{R}^d$. Stochastic differential equations are studied both for their distinctive properties, which set them apart from deterministic equations, and for their applications in mathematics, physics, finance, and other fields.

In this course, we plan to show some distinctive properties and tools of SDEs, as well as some examples and applications. In the following, we give some possible topics of the course, the precise list of topics can change according also to time restriction and to the students' preferences.

1. Well-posedness, chain rule, Feynman-Kac formulae: We will recall the classical existence and uniqueness result for solutions to SDEs and the stochastic chain rule, namely Itô formula. We will also recall the link between the SDEs and the second-order linear PDEs, namely the Feynman-Kac formulae.

2. Uniqueness in law and Girsanov theorem: Beside the classical pathwise uniqueness concept (two solutions X and Y satisfies $X = Y$ P-a.s.), there is another concept of uniqueness, namely uniqueness in law ($\mathrm{Law}(X) = \mathrm{Law}(Y)$). Remarkably, uniqueness in law is strictly weaker than pathwise uniqueness. Another properties of the law of the solution to an SDE is Girsanov theorem: in the simple setting of additive noise (i.e. $\sigma = \text{constant} \neq 0$), under suitable regularity assumptions, the law of the solution is absolutely continuous with respect to the Wiener measure, with an explicit density.

3. One-dimensional SDEs and Feller test of explosion: Given a one-dimensional ($d = 1$) SDE

$$\mathrm{d}X_t = \frac{1}{X_t} 1_{X_t \neq 0} \mathrm{d}t + c\,\mathrm{d}W$$

with $X_0 > 0$, does X ever touch zero with positive probability? without noise (i.e. $c = 0$) no, but with noise it can happen:

Theorem 0.1 (Informal). *For every $c > \sqrt{2}$, with positive probability there exists a (stopping) time τ with $X_\tau = 0$.*

More generally, using suitable tools for one-dimensional SDEs, we can characterize the behaviour of the solution near the singular points and at infinity.

4. Long-time behaviour and invariant distributions: An invariant distribution is a probability measure μ on $\mathbb{R}^d$ such that

$$\mathrm{Law}(X_0) = \mu \Rightarrow \mathrm{Law}(X_t) = \mu \quad \forall t \geq 0.$$

Invariant distributions are relevant at the physical level because they represent the state that the system reaches after a long time. For example, one has

Theorem 0.2 (Informal). *Consider the SDE*

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}W_t$$

for a smooth potential V with fast enough growth at infinity. Then there exists an invariant distribution of the form

$$\mu(\mathrm{d}x) = Z^{-1}e^{-2V(x)}\mathrm{d}x$$

with Z renormalization constant, and the law of X_t converges weakly, as $t \to +\infty$, to μ.

5. Flow properties: Calling X_t^x the solution with initial condition x, we want to study the regularity of X_t^x with respect to x. Due to technical reasons, the study of the regularity is more difficult than in the deterministic case, but it is relevant because it allows to solve some non trivial linear stochastic partial differential equations (transport equation).

6. Mean-field interacting article systems and McKean-Vlasov SDEs: Consider a system of N particles, where each particle is subject to an independent white noise and the particles interact through their empirical measure: precisely, consider

$$\mathrm{d}X_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} K(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \mathrm{d}W_t^i, \quad i = 1, \ldots N,$$

where $X_t^{i,N} \in \mathbb{R}^d$ represents the position of the i-th particle, W^i are independent Brownian motions and K is an interaction kernel. Such systems appear in (simplified) models in fluid dynamics (Euler equations), chemotaxis (Keller-Segel equations), neural networks (in the limit of infinite width or infinite depth) and other examples. What is the behaviour of the system for N large? In the limit $N \to \infty$, the behaviour of a typical particle can be described by one single SDE, where the drift depends on the law of the solution itself (McKean-Vlasov SDEs), namely

$$\begin{aligned} \mathrm{d}\bar{X}_t^i &= b(\bar{X}_t^i, \mathrm{Law}(\bar{X}_t^i))\mathrm{d}t + \mathrm{d}W_t^i, \\ b(x,\mu) &= \int_{\mathbb{R}^d} K(x-y)\mu(dy). \end{aligned}$$

Indeed, we have:

Theorem 0.3 (Informal)**.** *Assume that K is Lipschitz and bounded. For every i, we have*

$$\mathbb{E}[\sup_{t\in[0,T]} |X_t^{i,N} - \bar{X}_t^i|] \to 0 \quad \textit{as } N \to \infty.$$

7. Regularization by noise: An ordinary differential equation with non-smooth drift, for example the following one

$$\mathrm{d}X_t = \mathrm{sign}(X_t)\sqrt{|X_t|}\,\mathrm{d}t, \quad X_0 = 0,$$

can have multiple solutions. Surprisingly, the addition of a simple additive noise, i.e. $+\mathrm{d}W$ with W one-dimensional Brownian motion, restores uniqueness, even at a pathwise level!

For those who attend the course regularly, the exam consists of one or two seminars on assigned course topics to be held during class hours, and possibly some questions on course topics.

For those who do not attend the course, the exam consists of a traditional oral examination on the course topics.