Toric Varieties (Geometria algebrica F)

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Introduction

Toric varieties are a special class of algebraic varieties (i.e. spaces locally described as solutions to a system of polynomial equations), whose geometry is completely controlled by the combinatorics of the associated "fan", a finite collection of convex rational polyhedral cones in some real vector space $\mathbb{R}^n$, that overlap nicely. A single cone gives rise to an affine variety via the monoid algebra construction, and the position of the cones in the fan prescribes how to glue these affine pieces. Projective toric varieties can also be described via polytopes instead of fans.

Relevant examples are affine spaces $\mathbb{A}^n$ and projective spaces $\mathbb{P}^n$, their products, some of their blow-ups, etc. The name "toric" comes from the existence of an action on any such variety of an "algebraic torus", a product $\mathbb{G}_m^n$ of copies of the multiplicative group (over the field of complex numbers this is just $(\mathbb{C}^*)^n$ with component-wise multiplication), with a dense open orbit.

Being amenable to explicit computations, due to their connections to combinatorics and convex geometry, toric varieties are a very good testing ground for general conjectures in algebraic geometry. Since they were first introduced in the 70s, they have gathered a lot of interest over time, also thanks to applications in fields such as theoretical physics and applied mathematics, and are prominent in contemporary research. There are also important connections with topology, in particular with symplectic geometry.

Topics

The bulk of the course will consist in the basics of the theory of toric varieties, including: algebraic tori and their actions, affine, projective and general toric varieties, divisors/line bundles on toric varieties, the Cox ring construction. Some relevant basic notions in algebraic geometry will be briefly recalled and/or introduced, depending on the audience's familiarity with them. Time permitting, the latter part of the course will focus on one or more advanced topics (e.g. toric resolution of singularities, the classification of toric vector bundles, etc.), possibly depending on the students' interests.

Practical information

This is a 42 hours course, and will take place in the second semester. The language (English or Italian) will be decided together with the interested students at the start of the course.

The exam will consist of a 30-minute presentation on an advanced topic (to be decided with the instructor), followed by some questions about the material covered in the course.

Prerequisites are basics of commutative algebra and algebraic geometry (affine and (quasi-)projective varieties over an algebraically closed field, regular and rational functions and morphisms, local rings, smoothness, dimension, etc., all taught in the course "Elementi di geometria algebrica"). More advanced algebraic geometry (i.e. scheme theory) is not going to be required, although it would be helpful and make the course a more pleasant experience. Taking Vistoli's course at Scuola Normale (at least the first semester) is probably a good idea. There will be a preliminary meeting, close to the end of the first semester, with the students who are interested in the course, mostly aimed at gauging familiarity with this material. The course will be adjusted accordingly.

References

[1] Cox, David A.; Little, John B.; Schenck, Henry K., *Toric varieties.* Grad. Stud. Math., 124, American Mathematical Society, Providence, RI, 2011. xxiv+841 pp.

[2] Fulton, William, *Introduction to toric varieties.* Ann. of Math. Stud., 131, William Roever Lectures Geom., Princeton University Press, Princeton, NJ, 1993. xii+157 pp.