Course Description

Knot Theory A

(Teoria dei nodi A)

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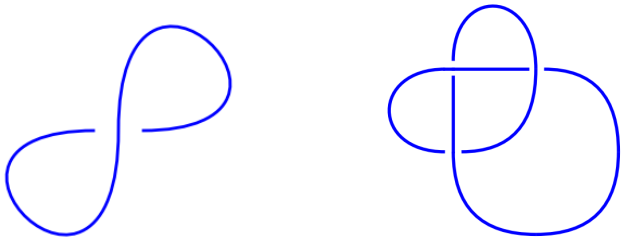
Introduction

Knot theory is the study of closed loops in 3-dimensional space. More formally, a knot $K \subset \mathbb{R}^3$ can be defined as a subset homeomorphic to a circle. Although this definition is not entirely precise for all purposes, it suffices as an introductory explanation and will be refined further in the course.

Two examples of knots are shown below:



Two knots K and K' are considered equivalent if there is a continuous family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $0 \leqslant t \leqslant 1$, such that $h_0 = \mathrm{id}_{\mathbb{R}^3}$ and $h_1(K) = K'$.

A knot invariant is a function from the set of knots to another set, whose value depends only on the equivalence class of the knot. The two knots illustrated above are not equivalent. To demonstrate this, one must construct a knot invariant that assigns different values to the two knots.

In the course, we will learn how to determine whether two knots are equivalent, how to define various knot invariants, and how to apply these invariants to the classification problem for knots.

Course Highlights

- **Reidemeister moves**. The two pictures above are examples of knot diagrams. Two such diagrams represent equivalent knots if and only if they are related by a sequence of local modifications, called *Reidemeister moves*.

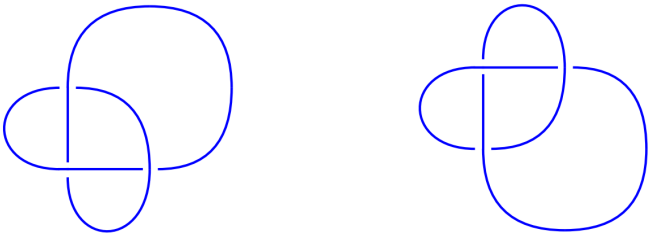
- **Seifert surfaces and knot genus**. Every knot $K \subset \mathbb{R}^3$ is the boundary of an embedded orientable surface $F \subset \mathbb{R}^3$, known as a *Seifert surface* for K. The minimal genus of a Seifert surface for K is an invariant of the knot, referred to as its *genus*. Up to knot equivalence, there is a unique knot of genus zero, known as the *unknot*. The genera of the two knots above are zero (left) and one (right), respectively, demonstrating that they are not equivalent.

- **Seifert form and Alexander polynomial**. Given a Seifert surface $F \subset \mathbb{R}^3$ for a knot $K = \partial F$, there exists a bilinear map $H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \longrightarrow \mathbb{Z}$ known as the *Seifert form*. If A is a matrix representing the Seifert form, the Laurent polynomial

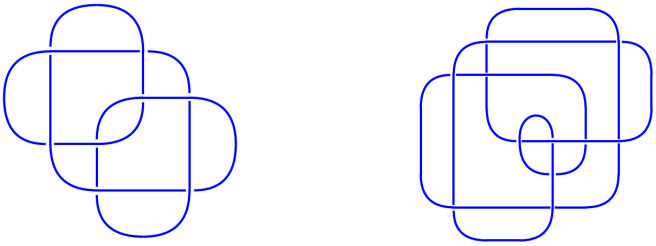
$$\Delta_K(x) = \det(xA - A^t) \in \mathbb{Z}[x, x^{-1}],$$

considered up to multiplication by $x^{\pm 1}$, is an invariant of the knot K called the *Alexander polynomial*. The two knots above can be distinguished by their Alexander polynomials.

- **Signature**. Given a matrix A representing the Seifert form of a knot K, one can define another invariant, called the *signature*. The signature is defined as the signature of the matrix $A - A^t$ and is denoted by $\sigma(K)$. The knots shown below are the *left-handed trefoil* (on the left) and the *right-handed trefoil* (on the right). They are mirror images of each other. While they share the same Alexander polynomial, they differ in their signatures.



- **Jones polynomial**. The *Jones polynomial* of a knot K is a Laurent polynomial invariant $V_K(t) \in \mathbb{Z}\left[t^{\frac{1}{2}}, t^{-\frac{1}{2}}\right]$. The following knots have the same Alexander polynomial and signature, but distinct Jones polynomials:



Extra information

- *Prerequisites:* Algebra of polynomials. Elements of algebraic topology (fundamental group, coverings, homology). Classification of surfaces.
- *Exam structure:* Oral exam. If teaching support is provided for the course, there will also be written assignments.
- *Textbooks:*

i. G. Burde and H. Zieschang, "Knots", De Gruyter, 1985.

ii. P.R. Cromwell, "Knots and Links", Cambridge U. Press, 2004.

iii. W.B. Lickorish, "An Introduction to Knot Theory", Springer, 1997.