224AA Ergodic Theory
(Teoria Ergodica)
First semester 2024-25
6 CFU, 42 hours

Instructor: Gianluigi Del Magno

1 Overview

This course is an introduction to Ergodic Theory, a part of dynamical systems theory that studies transformations on measure spaces using probabilistic methods. Ergodic theory investigates the long-term average behavior of systems as they evolve in time. Key concepts studied in this course include invariant measures, ergodicity and mixing.

Ergodic Theory originated from Boltzmann and Poincaré's work in the late 19th and early 20th centuries. Boltzmann's ergodic hypothesis stated that, over time, a system spends time in its states proportional to their phase space volume. Poincaré's recurrence theorem showed that certain systems eventually return to states near their initial ones. The field was formalized in the mid-20th century by mathematicians such as George D. Birkhoff and John von Neumann, who established the Ergodic Theorem and the Mean Ergodic Theorem, providing its mathematical foundation.

Ergodic theory has broad applications across various areas of mathematics. For example, it forms the foundation for understanding the long-term behavior of particle systems in statistical mechanics and random processes like Markov chains in probability theory, whereas in number theory, it aids in studying sequences and their distribution properties.

2 Course Description

This 42-hour course consists of three parts divided into 21 lectures. The first part covers the basic concepts and examples of measure-preserving transformations, ergodicity and mixing. The second part addresses the main

theorems in ergodic theory, including Birkhoff's Pointwise Ergodic Theorem and the Mean Ergodic Theorem. The third part explores advanced topics according to the interests of the students and the instructor, including spectral properties of dynamical systems, ergodic properties of some hyperbolic transformations and possibly applications to other areas of mathematics.

3 Course Outline

1. **Basic Concepts of Ergodic Theory**
    - Measure theory refresher
    - Invariant Measures
    - Poincaré Recurrence Theorem
    - Examples
2. **Ergodicity and Mixing**
    - Ergodicity
    - Unique ergodicity
    - Mixing
    - Examples
3. **Ergodic Theorems**
    - Mean Ergodic Theorem
    - Birkhoff's Pointwise Ergodic Theorem
    - Ergodic Decomposition
4. **Extra Topics**
    - Spectral Properties of Dynamical systems
    - Ergodic Properties of Expanding Maps
    - Applications

4 Bibliography

1. M. Viana and K. Oliveira, *Foundations of Ergodic Theory*, Cambridge University Press, 2016 (the e-book can be downloaded freely from the university library)

2. P. Walters, *An Introduction to Ergodic Theory*, Springer, 2000

3. M. Einsiedler and T. Ward, *Ergodic Theory with a View Towards Number Theory*, Springer, 2011

4. A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, 1995

5. M. Brin and G. Stuck, Introduction to dynamical systems, Cambridge University Press, 2002

6. Research articles on ergodic theory and applications

5 Assessment

The final grade is based on homework and a final oral examination.

The homework consists of a list of several problems. Students are allowed to work in groups, but the final solutions must be written individually and submitted before the last lecture of the course.

The oral examination consists of a 40-minute seminar on a topic agreed upon by the student and the instructor, followed by a 15-minute examination on the general theory presented in the course. A list of topics for the seminar will be provided by the instructor.

6 Prerequisites

Real Analysis, Measure Theory, Basic notions of Functional Analysis and Dynamical Systems.

7 Instructor

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