Algebraic combinatorics (Combinatoria algebrica)

2025/2026

(first semester)

Teacher: Michele D'Adderio

Practical informations

The course will last 42 hours, and it will take place in the first semester.

There will be an oral examination, consisting of a short seminar followed by questions about the content of the course.

Much of this course has virtually no prerequisites, and it can be attended by any student with genuine curiosity and open-mindedness.

After reding this document, whoever is interested in knowing more about this course can/should contact me via e-mail: michele.dadderio@unipi.it

Introduction

The most basic (and underestimated) mathematical activity is *counting*. In the past centuries, this primitive endeavor has led to sophisticated mathematical results, which grew into a still active field of research named *enumerative combinatorics*. It is not hard to imagine that this subject has applications in virtually every area of mathematics, but also in other disciplines, like computer science and physics. Indeed, it is often the case that to solve a (mathematical) problem at some point we need to count something.

But what is *algebraic combinatorics*? There are two possible answers, that often coexist: (1) the combinatorics emerging from objects and/or problems which are algebraic in nature, or (2) the use of algebraic tools to solve problems in (e.g. enumerative) combinatorics. Indeed, sometimes (in fact more interestingly) our (mathematical) problems involve integer numbers, that, a posteriori, turn out to be counting something: this is a strong indication that we are facing algebraic combinatorics!

The first part of this course will be an introduction to some fundamental theorems and methods commonly used in enumerative combinatorics. While we will certainly illustrate several points of view, our main focus will be the study of *generating functions*, certainly an algebraic, sometimes an analytic object. We will study important families of combinatorial objects and we will see several methods to solve a variety of enumerative problems.

After we developed our "combinatorial eye", in the second part of this course we will study mainly one of the fundamental combinatorial structures, that is ubiquitous in mathematics: partially ordered sets (a.k.a. *posets*). While going thorough their basic theory, we will see how to reinterpret under this new "lens" some of the problems and theorems discussed in the first part of the course. Sometimes this will provide a unifying point of view, often suggesting interesting generalizations. Developing the theory of posets, we will be naturally led to consider other fundamental combinatorial structures, like simplicial complexes, hyperplane arrangements and matroids.

In the spirit of the first part of the course, we will associate to these structures some interesting algebraic objects (typically polynomials, more generally formal power series), which encode useful information. These are often the focus of mainstream research in algebraic combinatorics today.

Along the way, we will indicate some of the many connections between these structures and other fields of mathematics, like representation theory, algebraic topology and algebraic geometry.

Content and goal

The nature of algebraic combinatorics does not allow to determine a "canonical syllabus" for a first course on the subject. This is reflected in the very different choices that different people made in the past: it is enough to intersect three or four syllabi of courses on the topic given around the world, or three or four tables of contents of dedicated books/lecture notes, to get the empty set.

Keeping this in mind, here is a rough (=flexible) list of objects and topics that we will/might cover: calculus of ordinary and exponential generating functions, exponential formula, Lagrange inversion, permutation statistics, symmetric and quasisymmetric functions, partially ordered sets (posets), Möbius inversion, simplicial complexes, hyperplane arrangements and matroids.

The main goal of this course is to develop a "combinatorial eye", and to learn how to recognize and deal with algebraic combinatorics when we encounter it.

WARNING!

While there are few courses in which (algebraic) combinatorics makes its appearance (mostly as a tool), in Pisa there is no introductory course on this beautiful subject. Hence, most likely this course might be your first experience where this kind of mathematics is the object of study (and not just a tool).

If you are reading this paragraph, probably you are not yet totally repulsed by this course. Since the nature of the topic is quite different from anything else you might have studied until now, I suggest you try to get an idea of what this is all about before deciding for/against it. Here are a few (not mutually exclusive) ways to do so:

(I) Skim thorough some/all items of the annotated list of references at the end of this document (some of them are free online).

(II) Surf through the website of the combinatorics group of Pisa:
`https://cult.dm.unipi.it/`
Here you can find a lot of information about the combinatorics that will appear in the course, but also about the related activities organized in the department.

(III) Attend the first few lectures. Notice that, because of the aforementioned lack of introductory courses in combinatorics here in Pisa, I prefer to keep the first few lectures quite basic. So you might have seen some of the covered results. Needless to say, the level of difficulty of the course will grow, and it will grow pretty fast. So keep this in mind...

(IV) Talk to me (`michele.dadderio@unipi.it`), and/or to anyone in the combinatorics group of Pisa (`https://cult.dm.unipi.it/members/`).

(V) Talk to your fellow students that got involved in some of the activities of the combinatorics group of Pisa (courses, reading seminars, research seminars, winter and summer schools, conferences, etc.), keeping in mind that their reliability is no better than the one of any review system... :-\

Annotated list of possible references

While I will not follow any specific reference, I provide here a list of resources, with some comments. Notice that (k) is beautifully written for every $k \geq 1$.

(0) e-learning webpage of my course “Combinatoria algebrica 2022-23”
[Here you can find notes from a previous (2022-2023) incarnation of this course. WARNING: the 2025-2026 version of this course will differ by at least 50% !!]

(1) Aigner, “A Course in Enumeration”, Springer

(2) Beck, Sanyal, “Combinatorial reciprocity theorems”, AMS
free online: `https://matthbeck.github.io/papers/crt.pdf`
[a geometric view (which we will not cover) on some results that we might cover]

(3) Eğecioğlu, Garsia, “Lessons in Enumerative Combinatorics”, Springer
[inimitable style]

(4) Martin, “Lecture Notes on Algebraic Combinatorics”, free online:
`http://jlmartin.ku.edu/CombinatoricsNotes.pdf`

(5) Mendes, Remmel, “Counting with Symmetric Functions”, Springer

(6) Sagan, “Combinatorics: The Art of Counting”, AMS
free online: `https://users.math.msu.edu/users/bsagan/`

(7) Stanley, “Enumerative Combinatorics. Volumes 1 and 2”, Cambridge
[the “Bible” of the subject, a masterpiece!]

(8) Stanley, “Algebraic Combinatorics. Walks, Trees, Tableaux, and More”, Springer
[easier for students than the previous one]

(9) Stanley, “An Introduction to Hyperplane Arrangements”, free online:
`https://math.mit.edu/~rstan/arrangements/arr.html`
[a classic reference]

(10) Wachs, “Poset Topology: Tools and Applications”, AMS
free online: `https://www.math.miami.edu/~wachs/papers/toolnotes.pdf`

(11) Wilf, “Generatingfunctionology”, AK Peters/CRC
[another classic]