Elements of Calculus in homogeneous groups (Elementi di Calcolo in gruppi omogenei)

a.y. 2025/2026
second semester

Teacher: Valentino Magnani

A description of the course

The present course offers an introduction to the analysis and geometry of some "non-Riemannian spaces" modeled on a specific class of nilpotent Lie groups. Among them, the Heisenberg group plays a prominent role, and its rich geometry will be explored, starting from its algebraic and analytic representations, and then looking at its connections to complex analysis, quantum mechanics, and contact geometry. The course will first provide a general understanding of real Lie groups, Lie algebras, and several basic facts, presenting Lie's theorems, the exponential mapping, and the Baker-Campbell-Hausdorff-Dynkin formula. Gradations and stratifications of nilpotent Lie groups will be introduced and the important classes of homogeneous and stratified groups will be studied, with special emphasis on their different homogeneous distances, the sub-Riemannian distance, Haar measure and the Hausdorff measure. A notion of differentiability for these special noncommutative Lie groups leads to a general version of the implicit function theorem and of the inverse mapping theorem. The rich differentiable and metric-measure structure of these groups also allows us to consider a generalized version of the Laplacian, called sub-Laplacian, the associated harmonic functions, Sobolev inequalities and Poincaré inequalities with respect to their noncommutative algebraic structure. The final part of the course introduces students to the question of computing the intrinsic area of smooth submanifolds in homogeneous groups, introducing the first elements of Geometric Measure Theory in homogeneous groups, also drawing some connections to intrinsic differential forms.

The course aims to build a bridge between classical topics about analysis and geometry on Lie groups and some aspects of current mathematical research. It provides connections among algebraic, analytic and geometric properties of homogeneous groups, illustrating how algebraic structures can be connected to analytic and geometric tools, including PDEs. For students, the course is an opportunity to explore a rapidly developing area of research, while also offering a general background for broader interdisciplinary applications.

About the final

The final consists of an oral examination divided into two parts. Concerning the first part, the student will be examined on the first three sections of the syllabus. The second part requires the student to present a seminar that delves into one specific topic that has been treated during the course. The seminar is based on a written work that will be submitted to the teacher prior to the final.

Detailed program of the course

I. The Heisenberg Group. Various representations of the Heisenberg group and its connections with complex analysis, quantum mechanics, and contact manifolds. Algebraic and metric structure of the Heisenberg group.

II. Real Lie Groups and Lie Algebras. Real Lie groups and Lie algebras, with brief remarks on the infinite-dimensional case. The exponential map and its differential. Matrix groups. The Baker–Campbell– Hausdorff formula and Dynkin's formula. Homomorphisms of Lie groups and Lie algebras. The three fundamental theorems of Lie theory. Real-analyticity of the exponential map for connected, simply connected, and nilpotent Lie groups.

III. Gradings and Homogeneous Lie Groups. Gradings of Lie algebras and homogeneous Lie groups. Construction of homogeneous and sub-Riemannian distances. Graded, stratified, and Carnot Lie groups. Homogeneous groups and their universal representations. Graded coordinates, representation of the group operation, and left-invariant vector fields. Haar measure on locally compact Lie groups. Bi-invariance of the Lebesgue measure in nilpotent groups. H-type groups and the Cygan norm.

IV. Sub-Laplacian and Harmonic Functions. The sub-Laplacian and its harmonic functions. Fundamental solution of the sub-Laplacian. Intrinsic Sobolev spaces and the Poincaré inequality.

V. Differential Calculus on stratified Lie Groups. Differential calculus associated with the group operation. Mean value theorem, inverse function theorem, rank theorem, and implicit function theorem on stratified groups.

VI. Hausdorff measure and an introduction to Geometric Measure Theory in homogeneous groups. Properties of the Hausdorff measure in homogeneous groups and relationships among Hausdorff, Haar, and Lebesgue measures. Left-invariant differential forms and weights of k-forms and of k-vector fields. Intrinsic area measure, differentiability of measures in metric spaces, spherical factor, the intrinsic area problem for surfaces, surface weights, and their relation to the Hausdorff dimension with respect to the homogeneous distance.

Practical information

The course will be held in Italian, unless some of the students need the lectures to be taught in English. The overall duration of the course is 42 hours and will take place in the second semester. Additional information is available at the webpages:

`https://unipi.coursecatalogue.cineca.it/insegnamenti/2025/53083_707407_70557/2025/53084/11514?coorte=2025&schemaid=9445`

`http://people.dm.unipi.it/magnani/Corsi/ECGO/ECGO.html`

All students who have questions about this course are invited to write me at valentino.magnani@unipi.it