Geometria Riemanniana – Riemannian Geometry

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1 Contents

The course will focus on comparison theorems in Riemannian geometry, namely on results that put constraints on the topology of Riemannian manifolds that satisfy various bounds on the sectional and Ricci curvature. Most of the results are contained in the classical book of Cheeger and Ebin [2], but I will also refer to the more recent book by Petersen [1].

I will begin with some explicit curvature computations, particularly for the Berger sphere and the Fubini-Study metric. I will then introduce variations, and in particular geodesic variations, i.e. variations of a geodesic through geodesics. At the infinitesimal level, this leads to the notion of Jacobi vector fields, which will be a central topic in the course. A first application of Jacobi fields will be the classification of manifolds of constant curvature through the Cartan-Ambrose-Hicks theorem.

On the other hand, Jacobi fields allow one to characterize the cut locus (roughly, the locus in the tangent space T_pM at which $\exp_p$ ceases to be a diffeomorphism). In particular, we will see that for manifolds with sectional curvature $\sec \leq 0$, the exponential map has no critical points, leading to the Cartan-Hadamard theorem, stating that any complete manifold with $\sec \leq 0$ is topologically a quotient of $\mathbb{R}^n$. We will also see two further restrictions on the topology (more precisely, the fundamental group) in this context, namely the theorems of Cartan and Preissmann.

In the positively curved setting, I will illustrate Myers' theorem for complete metrics on which the Ricci curvature Ric is positively bounded below, stating in particular that such manifolds are compact, and the Cheeger-Gromoll splitting theorem, which holds more generally for $\mathrm{Ric} \geq 0$.

Turning to the more restrictive case where $\sec > 0$, I will prove Synge's theorem, based on the energy variation formula, and two results where sec is additionally required to be bounded above, namely Klingerberg's estimate on the injectivity radius and Berger's sphere theorem (at least in even dimensions, following [3]).

I will conclude with the soul theorem of Cheeger and Gromoll, which realizes any noncompact complete manifold with $\sec \geq 0$ as the normal bundle of a totally geodesic compact submanifold.

2 Practical information

This course takes off where “Istituzioni di Geometria” ends. It is advisable to follow “Geometria e Topologia differenziale” first, then “Istituzioni di Geometria”, before taking Riemannian geometry.

Failing that, the students will have to fill in some prerequisites on their own. The following will be assumed as prior knowledge:

- smooth manifolds and fiber bundles;
- Riemannian metrics; Levi-Civita-connection, curvature;
- vector fields, Lie derivative, flow of a vector field;
- geodesics and exponential map, normal coordinates, Hopf-Rinow theorem.

I do not expect to deviate from the indicated bibliography. Lecture notes (in English), updated regularly to reflect the topics covered in class, will be made available on the Teams page of the course.

References

[1] Petersen, Riemannian Geometry. Springer.

[2] Cheeger-Ebin. Comparison theorems in Riemannian geometry. AMS Chelsea publishing.

[3] Do Carmo, Riemannian geometry. Birkhäuser.