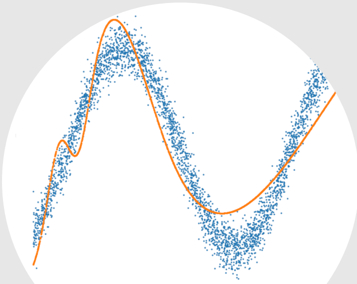


Istituzioni di Analisi Numerica

IAN

Instructor Info

Paola Boito
Fabio Durastante

Office hours by appointment

paola.boito@unipi.it

fabio.durastante@unipi.it

Course Info

Prerequisites: Analisi Numerica

3 lessons per week

2 hour lectures/72 h

11 CFU

Objectives

Acquire theoretical and computational skills in the theory of function approximation, orthogonal polynomials, approximate integration, matrix functions, and numerical resolution of partial differential equations using finite difference methods.

Course Content

Orthogonal Polynomials

We recall the definition of orthogonal polynomials. Let P be the set of polynomials with real coefficients. Let $\omega(x) : [a, b] \to \mathbb{R} \cup \{\infty\}$ such that, for $a < x < b$, $\omega(x) \in \mathbb{R}$, $\omega(x) > 0$, and $\int_a^b f(x)\omega(x)dx$ exists and is finite for every polynomial $f(x)$. We define the inner product on P as:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)dx.$$

A set of polynomials $\{p_i(x)\}_{i=0,1,\ldots}$ such that $\deg(p_i) = i$ and $\langle p_i, p_j\rangle = 0$ if $i \neq j$ is called a set of *orthogonal polynomials*.

An example is the Chebyshev polynomials of the first kind, which are obtained with $[a, b] = [-1, 1]$ and $\omega(x) = 1/\sqrt{1-x^2}$.

We will study various properties, useful from both a theoretical and computational perspective: properties of the zeros, three-term recurrences, relationships with tridiagonal matrices, etc.

These polynomials will be an important ingredient in the subsequent topics of the course.

Numerical Integration

Given $f \in C([a, b])$, we want to approximate $S[f] = \int_a^b f(x)dx$ with a formula of the type:

$$S_{n+1}[f] = \sum_{i=0}^{n} w_i f(x_i),$$

where $a \leq x_0 < x_1 < \ldots < x_n \leq b$ and w_i, $i = 0, \ldots, n$, are given weights.

Interpolation formulas based on the zeros of appropriate orthogonal polynomials (Clenshaw-Curtis and Gaussian formulas) will be studied.

Approximation of Continuous Functions

Linear Approximation

Given a set of linearly independent functions

$$\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\},$$

a function $f \in C([a, b])$, and a norm $\|\cdot\|$, we aim to determine coefficients α_i^*, $i = 0, 1, \ldots, n$, such that the approximation

$$g_n(x) = \sum_{i=0}^{n} \alpha_i^* \phi_i(x)$$

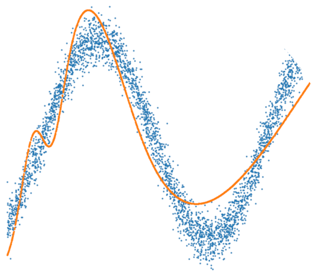
satisfies

$$\|f(x) - g_n(x)\| = \min_{\alpha_0,\ldots,\alpha_n} \left\| f(x) - \sum_{i=0}^{n} \alpha_i \phi_i(x) \right\|.$$

We will establish the existence of such a solution and analyze its theoretical and computational properties, with particular emphasis on the 2-norm and the ∞-norm. In the latter case, the problem reduces to the following: given $n \in \mathbb{N}$ and $f \in C([a, b])$, find a polynomial p of degree at most n that minimizes the uniform distance

$$\|f - p\|_{\infty, C[a,b]}.$$

In particular, we will show that this polynomial exists, is unique, and is characterized by an equioscillation property (Chebyshev's equioscillation theorem).



Istituzioni di Analisi Numerica

IAN

Instructor Info

Paola Boito
Fabio Durastante

Office hours by appointment

paola.boito@unipi.it

fabio.durastante@unipi.it

Course Info

Prerequisites: Analisi Numerica

3 lessons per week

2 hour lectures/72 h

11 CFU

Splines and Other Approximations

We will present other types of approximations, such as piecewise polynomial approximations and rational approximations (e.g., Padé approximants). The case of cubic splines will be discussed in detail: given $f \in C[a,b]$ and nodes $a = x_0 < x_1 < \ldots < x_n = b$, we want to determine a function $s \in C^2[a,b]$ such that its restrictions to $[x_i, x_{i+1}]$ coincide with polynomials of degree (at most) 3 for $i = 0, \ldots, n-1$, and

$$s(x_i) = f(x_i), \qquad i = 0, \ldots, n,$$

plus additional conditions.

In particular, we will see how the computation of cubic splines translates to a linear algebra problem with favourable computational properties.

Matrix Functions

A classic matrix function is the exponential, defined as:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!},$$

where $A \in \mathbb{C}^{n \times n}$. A general definition of a matrix function $f(A)$ will be introduced, starting from a function $f : \mathbb{C} \to \mathbb{C}$, which satisfies certain regularity properties. Some theoretical and computational properties of matrix functions will be studied.

Numerical Treatment of PDEs

We will discuss the numerical solution of PDEs via the finite difference method, with applications to the Poisson problem, the heat equation, and the wave equation. The stability of such approaches will be discussed, together with convergence analysis, following a mostly matrix-theoretic approach.

Reading material

Useful readings include the following works

- R. Bevilacqua, D.A. Bini, M. Capovani, O. Menchi, Metodi Numerici, Zanichelli, 1992
- D.A. Bini, M. Capovani, O. Menchi, "Metodi numerici per l'algebra lineare", Zanichelli, 1988.
- Eugene Isaacson and Herbert Bishop Keller, Analysis of Numerical Methods. John Wiley & Sons, Inc., New York, 1966.
- R.J. LeVeque. Finite Difference Methods for Ordinary and Partial Differential Equations. SIAM 2007.
- W. Rudin, Real and Complex Analysis, Second Edition, Tata McGraw-Hill, 1974.
- J. Stoer, R. Burlisch, Introduction to Numerical Analysis, Third Edition, Springer, 2002.

Lecture notes supplied by the lecturers are available from the course's web page.

Exam method

The exam is divided into two parts.

- Final written exam
- Final oral exam