0062A Random Dynamical Systems
(Sistemi Dinamici Aleatori)
Second semester 2025-26
6 CFU, 42 hours

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1 Overview

Mathematical models are often designed to describe how real-world processes evolve over time. In reality, however, these processes are constantly affected by unpredictable factors. Since these effects cannot usually be described exactly, it is natural to model them as random perturbations of deterministic systems.

The evolution of a deterministic dynamical system is obtained by repeatedly applying the same rule. By contrast, the evolution of a random dynamical system (RDS) is obtained by applying rules that vary randomly at each step. The behavior of the system is therefore the result of the interplay between the deterministic rules and randomness.

The purpose of this course is to study the long-term behavior of systems whose evolution is governed by randomly chosen transformations. The main tools come from ergodic theory, with a special emphasis on a functional-analytic approach based on transfer operators.

2 Course Description

The general setup is as follows. Let $\mathcal{F}$ be a collection of transformations $f : X \to X$ acting on a space X. A random map is a rule that randomly selects one transformation from $\mathcal{F}$ at each step, according to a probability distribution μ on $\mathcal{F}$.

Now consider an infinite sequence of random maps $\{\phi_1, \phi_2, \phi_3, \dots\}$, all chosen independently according to the same distribution μ. At step n, the state of the system is obtained by applying all the maps chosen so far:

$$S_n = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1.$$

The map S_n itself is random. If μ always selects the same map f, the system becomes deterministic and evolves just by iterating f: $S_n = f^n$.

This framework covers many interesting situations. For example: if $\mathcal{F}$ consists of a single map, one recovers a deterministic dynamical system; if $\mathcal{F}$ is a set of linear operators, one obtains products of random matrices; if $\mathcal{F}$ comes from stochastic flows, the framework can describe solutions of stochastic differential equations.

The course is divided into two parts. The first part covers the foundations of the theory. We begin with a review of essential concepts from deterministic dynamical systems, including invariant measures and ergodicity. We then introduce the framework of random dynamical systems (RDS), presenting their formulation via cocycles and skew-product representations. Connections with Markov chains will be highlighted, as well as the role of stationary measures in describing long-term statistical behavior.

The second part of the course focuses on applications, showing how random dynamical systems appear in different areas of mathematics and applications.

A key tool in this course is the transfer operator, which describes how probability measures evolve under the dynamics and provides a rigorous foundation for the ensemble approach of statistical physics. Under suitable assumptions, these operators exhibit strong functional-analytic properties, which allow one to derive quantitative results on stationary measures and their robustness under perturbations. From these properties, we will deduce consequences with significant practical applications, such as: linear response of observable averages and related control problems, statistical estimates for the distribution of extreme events, and reliable numerical methods for studying random dynamical systems.

3 Course Outline

The following is a tentative list of topics to be covered in the course:

1. Preliminaries on ergodic theory of deterministic maps: a) invariant measure of a deterministic map, b) ergodic theorems, c) ergodicity

2. Random transformations

3. Stationary measures
4. Ergodic stationary measures
5. Spectral properties of random dynamical systems: Koopman and Transfer operators
6. Spectral gap
7. Applications to random dynamical system with additive noise
8. Linear response and extreme events

4 Bibliography

1. M. Viana, Lectures on Lyapunov exponents, Cambridge Stud. Adv. Math., 145, Cambridge University Press, 2014
2. L. Arnold, Random Dynamical Systems, Springer Monogr. Math, Springer, 1998
3. Y. Kifer, Ergodic theory of random transformations, Progr. Probab. Statist., 10, Birkhäuser, 1986
4. A. Furman, Random walks on groups and random transformations. Handbook of dynamical systems, Vol. 1A, 931–1014, North-Holland, Amsterdam, 2002
5. S. Galatolo, Statistical properties of dynamics. Introduction to the functional analytic approach, arXiv:1510.0261

5 Assessment

For attending students, the assesment will consist of a 40-minute seminar on a topic agreed upon by the student and the instructor followed by a 15-minute examination on the general theory presented in the course. A list of topics for the seminar will be provided by the instructor. For non-attending students, the assessment will consist of a full oral examination on the entire content of the course.

6 Prerequisites

Measure-theoretic probability, functional analysis, and basic ergodic theory (or willingness to acquire them during the course; the initial lectures provide a reviews the fundamental notions of ergodic theory).

7 Instructors

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