ANALISI MATEMATICA B
LEZIONE 25 - 20.11.2020

$\nearrow$ $\frac{1}{2}$ è punto limite per $a_n = \sqrt{n} - \lfloor\sqrt{n}\rfloor$
Trovare n_k : $a_{n_k} \to \frac{1}{2}$

Punti limite: ℓ è un punto limite di $f(x)$ per $x \to x_0$ se

(i) $\forall U$ intorno di ℓ $f(x) \in U$ freq. per $x \to x_0$

o equivalentemente (ii) $\exists\, a_n \to x_0$, $a_n \neq x_0$, $f(a_n) \to \ell$.

funzioni	successioni
$f(x)$	a_n
$x \to x_0$	$n \to +\infty$
$a_n \to x_0$ $a_n \neq x_0$	n_k, $n_k \to +\infty$ (*) n_k strett. crescente
$f(a_n) \to \ell$	$a_{n_k} \to \ell$

Se L è l'insieme dei punti limite per $x \to x_0$

$$\limsup_{x\to x_0} f(x) = \sup L$$

$$\liminf_{x\to x_0} f(x) = \inf L$$

$0 \quad \frac{1}{3}$

$a_n = (-1)^n \qquad 1, -1, 1, -1, 1, -1 \ldots \rightsquigarrow 0$

non ha limite

<u>Convergenza alla Cesàro</u>:

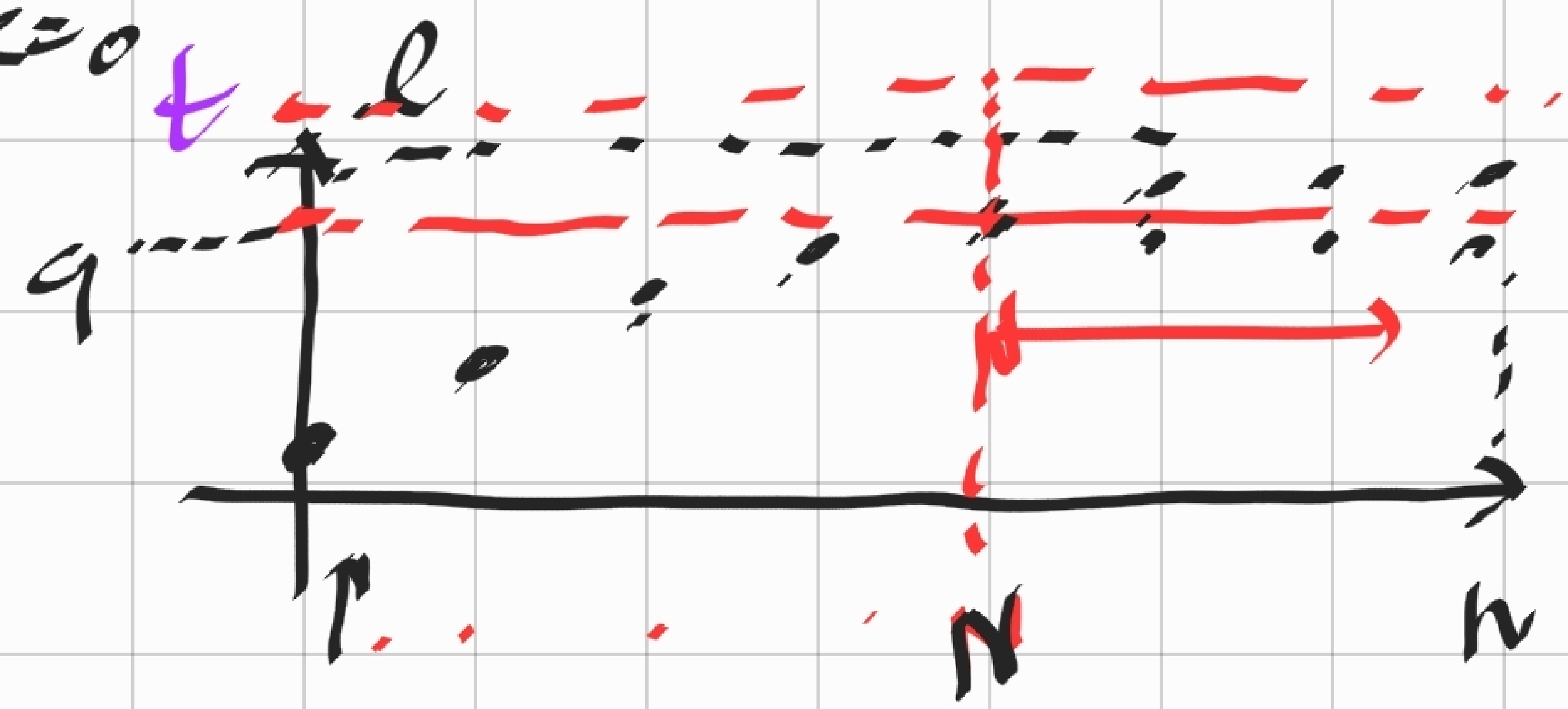
a_n converge a ℓ secondo Cesàro se

$$\frac{1}{n}\sum_{k=1}^{n} a_k \longrightarrow \ell$$

<u>Teorema</u> se $a_n \to \ell$ allora $\qquad \ell \in \overline{\mathbb{R}}$

$$b_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k \longrightarrow \ell \qquad \square$$

<u>dim</u>



Preso $q < \ell$ $\exists N$ t.c. $a_k \geq q$, $\forall k > N$ $\quad (\leq t)$

$t > \ell$

$$b_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k = \frac{1}{n}\sum_{k=0}^{N-1} a_k + \frac{1}{n}\sum_{k=N}^{n-1} a_k$$

$$\underset{(\leq)}{\geq} \frac{1}{n}\sum_{k=0}^{N-1} a_k + \frac{n-N}{n}\cdot q \ (t) = c_n \quad \forall n > N$$

$0 \qquad C \qquad 1$

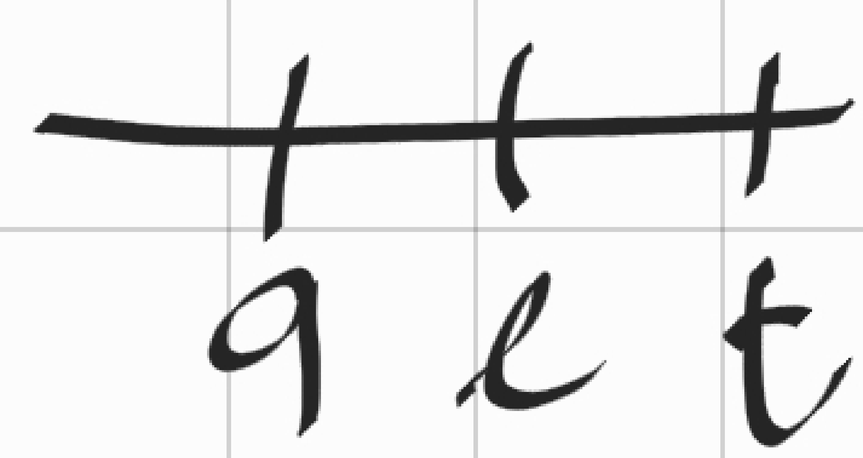
$n \to +\infty$

$$\longrightarrow q$$

$$\begin{bmatrix} b_n \geq c_n \to q \\ b_{n_k} \geq c_{n_k} \to q \\ b_{n_k} \to x \\ x \geq q \end{bmatrix}$$

$\forall q < \ell : \liminf\limits_{n \to +\infty} b_n \geq q$

$\forall t > \ell : \limsup\limits_{n \to +\infty} b_n \leq t$



$$\begin{cases} \liminf\limits_{n \to +\infty} b_n \geq \ell \\ \limsup\limits_{n \to +\infty} b_n \leq \ell \end{cases}$$

$$\ell \le \liminf \le \limsup \le \ell$$

$$\liminf b_n = \limsup b_n = \ell$$

$$L = \{\ell\}$$

$\lim b_n$ esiste! $\lim\limits_{n\to+\infty} b_n = \ell$ $\square$

Corollario Se $\dfrac{a_{n+1}}{a_n} \to \ell$

$a_n > 0$

Allora $\sqrt[n]{a_n} \to \ell$ $\square$

(Teorema (criterio della radice)

$\sqrt[n]{a_n} \to \ell < 1 \Rightarrow a_n \to 0$)

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dim (corollario) $\qquad \dfrac{a_{n+1}}{a_n} \to l$

$$\sqrt[n]{a_n} = \sqrt[n]{a_0 \cdot \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdots\cdots \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}}}$$

$$\ln \sqrt[n]{a_n} = \frac{1}{n}\left[\ln a_0 + \ln \frac{a_1}{a_0} + \dots + \ln \frac{a_n}{a_{n-1}}\right]$$

$$= \underbrace{\cancel{\frac{1}{n}} \ln a_0}_{\to 0} + \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} \ln \frac{a_{k+1}}{a_k}}_{\downarrow \text{ Cesàro}}$$

per $n \to +\infty$

$\ln l$

$$\Rightarrow \sqrt[n]{a_n} \to l. \qquad \square$$

Esercizio $\quad \lim\limits_{n\to+\infty} \dfrac{n}{\sqrt[n]{n!}} = e$

$$= \lim_{n\to+\infty} \frac{\sqrt[n]{n^n}}{\sqrt[n]{n!}}$$

$$a_n = \frac{n^n}{(n+1)!} \qquad \frac{a_{n+1}}{a_n} \to e \qquad \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} \qquad x^y$$

a^x $\quad a>0$, $x \in \mathbb{R}$

a^x $\quad a \in \mathbb{R}$, $x \in \mathbb{N}$

$$n! \ll n^n$$

$\cancel{\Downarrow}$ No!

$$\sqrt[n]{n!} \ll \sqrt[n]{n^n}$$

$$a_n^{b_n} = e^{b_n \ln a_n}$$

$$0^0 = e^{0\cdot(-\infty)}$$

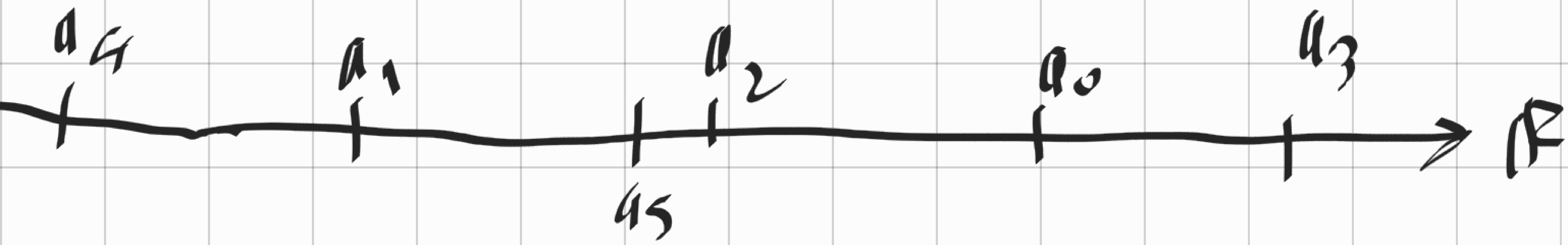
F.I.

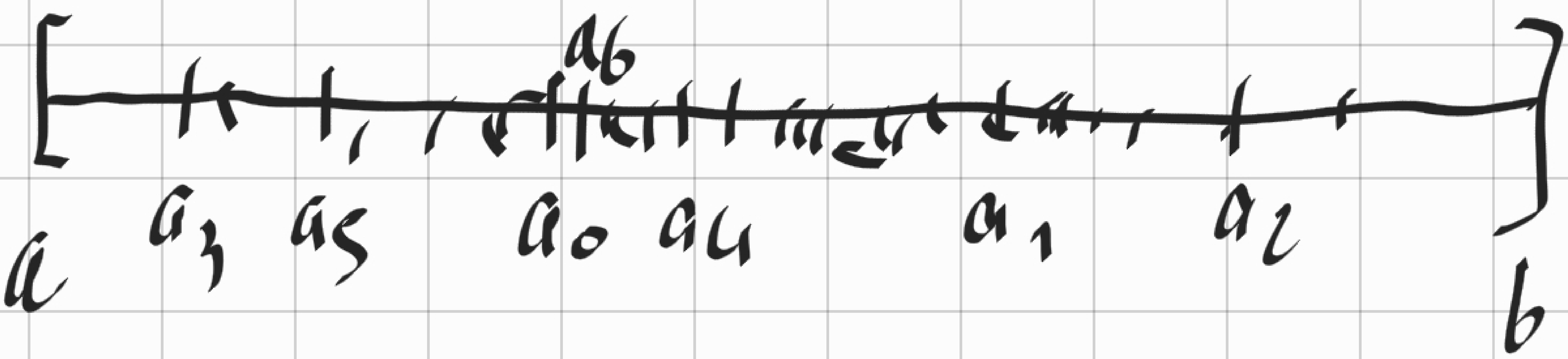
Teorema di Bolzano-Weierstraß

Se a_n è una successione qualunque esiste una estratta a_{n_k} regolare. (1)

Più precisamente se a_n è limitata allora esiste una estratta a_{n_k} convergente. (2)

Se a_n non è limitata esiste una estratta divergente. (3)





Es $\quad a_n = (-1)^n \cdot n$

$0, 1, -1, 2, -2, 3, -3 \ldots\ldots$

Es $\quad a_n = \sqrt{n+1} - \lfloor \sqrt{n} \rfloor$

è limitata:

$$0 \le a_n \le \sqrt{n+1} - (\sqrt{n}-1) \le 2$$

$$\sqrt{n+1} = \sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right) \qquad +1$$

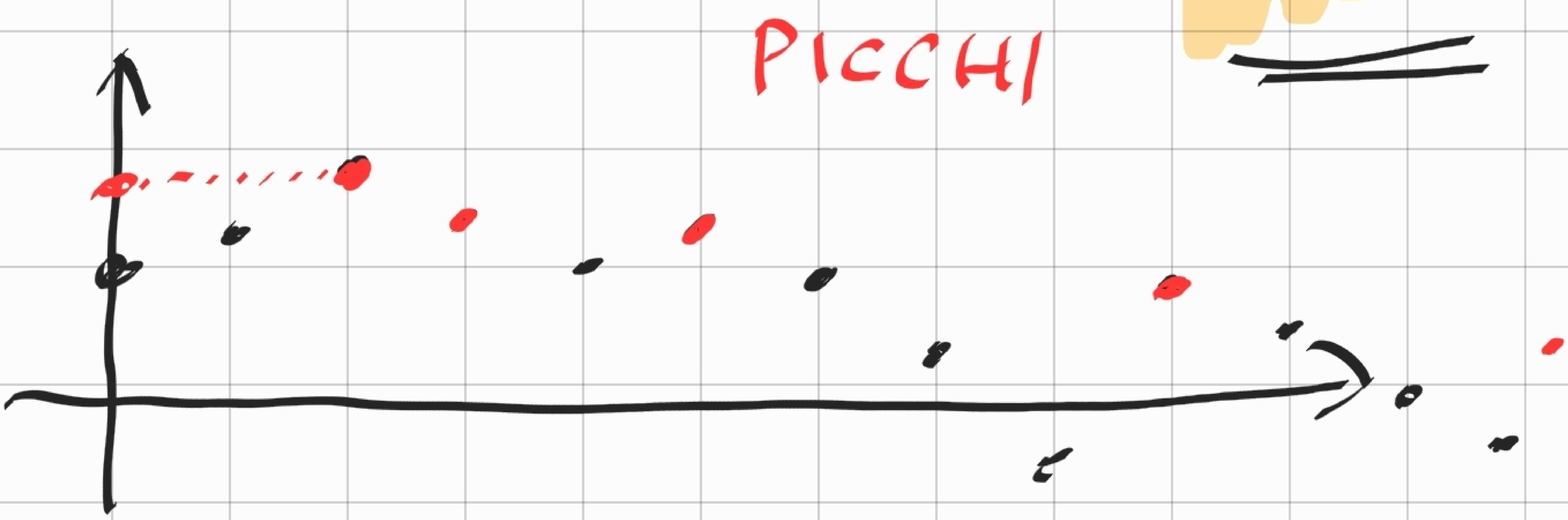
$$\sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \le \frac{1}{\sqrt{2}+1} \le 1$$

esiste una estratta convergente

$$a_{n_k} \to l \qquad 0 \le l \le 2$$

dim (più forte)

Data a_n esiste a_{n_k} monotona.



Dico che a_n è un picco

se $a_n \ge a_k$ se $k > n$

$$P = \{ n \in \mathbb{N} : a_n \text{ picco} \}$$

$$= \{ n \in \mathbb{N} : k > n \Rightarrow a_k \le a_n \}$$

(1) se P è infinito

$$P = \{ n_0, n_1, n_2, \dots, n_k, \dots \}$$

$$n_0 = \min P$$

$$n_1 = \min P \setminus \{n_0\}$$

$$\vdots$$

$$n_{k+1} = \min \underbrace{P \setminus \{n_0, n_1, \dots, n_k\}}_{\neq \emptyset}$$

a_{n_k} è decrescente

$$\underline{n_{k+1} > n_k} \qquad n_k \in P$$

$$\Downarrow \quad a_{n_k} \text{ picco}$$

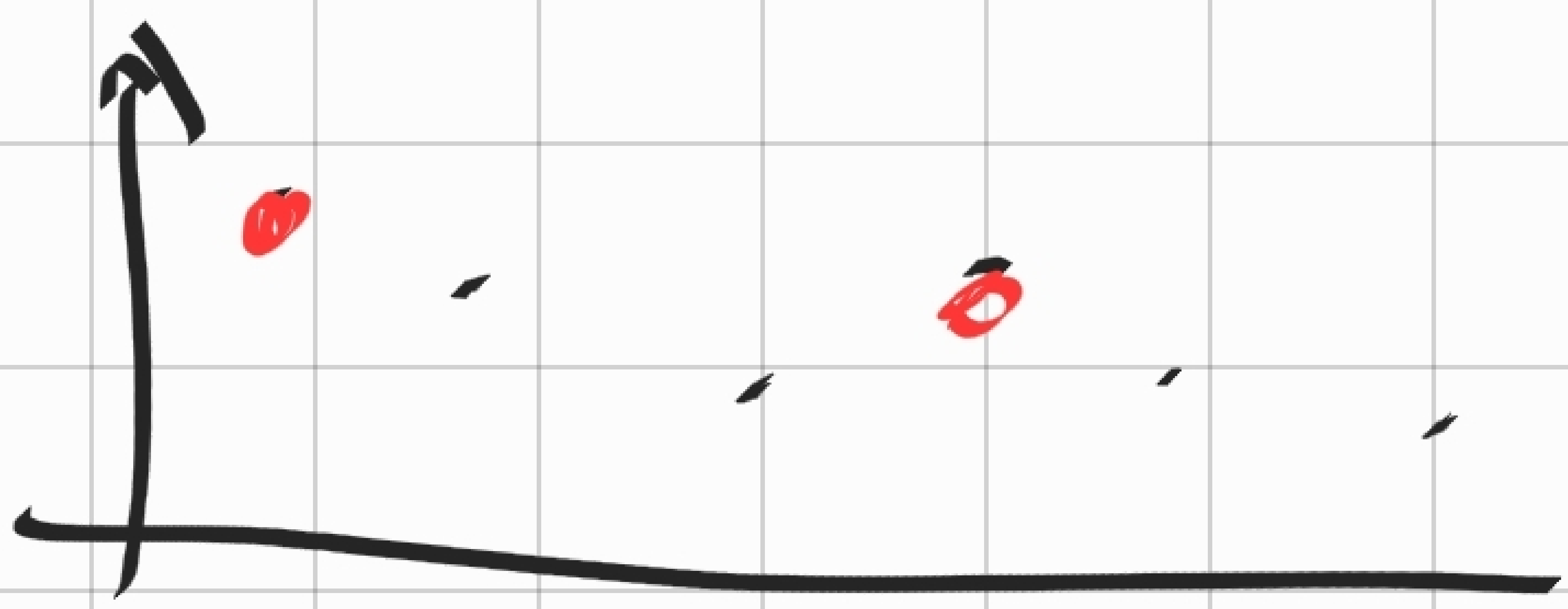
$$a_{n_{k+1}} \leq a_{n_k}$$

a_{n_k} è decrescente ✓

(2) se P è finito

$P = \{p_0, p_1, \dots, p_N\}$

$N = \max P$



Se $n > N$ $\quad a_n$ non è un picco

$\forall n > N \ \exists k > n : a_k > a_n$

$\to n_0 = N+1$

$n_1 : \exists n_1 > n_0 \quad a_{n_1} > a_{n_0}$

$n_2 : n_2 > n_1 \quad a_{n_2} > a_{n_1}$

⋮

$$a_{n_{k+1}} > a_{n_k} \quad \neq \emptyset$$

$$n_{k+1} = \min \{ n > n_k : \underline{\underline{a_n > a_{n_k}}} \}$$

a_{n_k} è <u>strettamente crescente</u>

$\square$

<u>B-W</u>: $\lim a_{n_k}$ esiste

(per monotonia)

(<u>oss</u> se $\sup a_n = +\infty$

(a_n è superiormente illimitata)

(~~allora $P = \emptyset$~~

~~a_{n_k} strett. crescente~~)

$\square$

$n_0 = 0$ $\qquad$ n_k t.c. $a_{n_k} \geq k$

$n_1 > n_0$ t.c. $a_{n_1} \geq 1$

$n_2 > n_1$ t.c.

$\Leftarrow \left(a_{n_2} > \max \{2, a_0, a_1, a_2, \dots, a_{n_1}\} \right)$

$\Downarrow$

$a_{n_2} > n_2$

n_k t.c. $a_{n_k} > \max\{k, \underbrace{a_0, a_1 \dots}, a_{n_{k-1}}\}$ $\square$

$n_k > n_{k-1}$

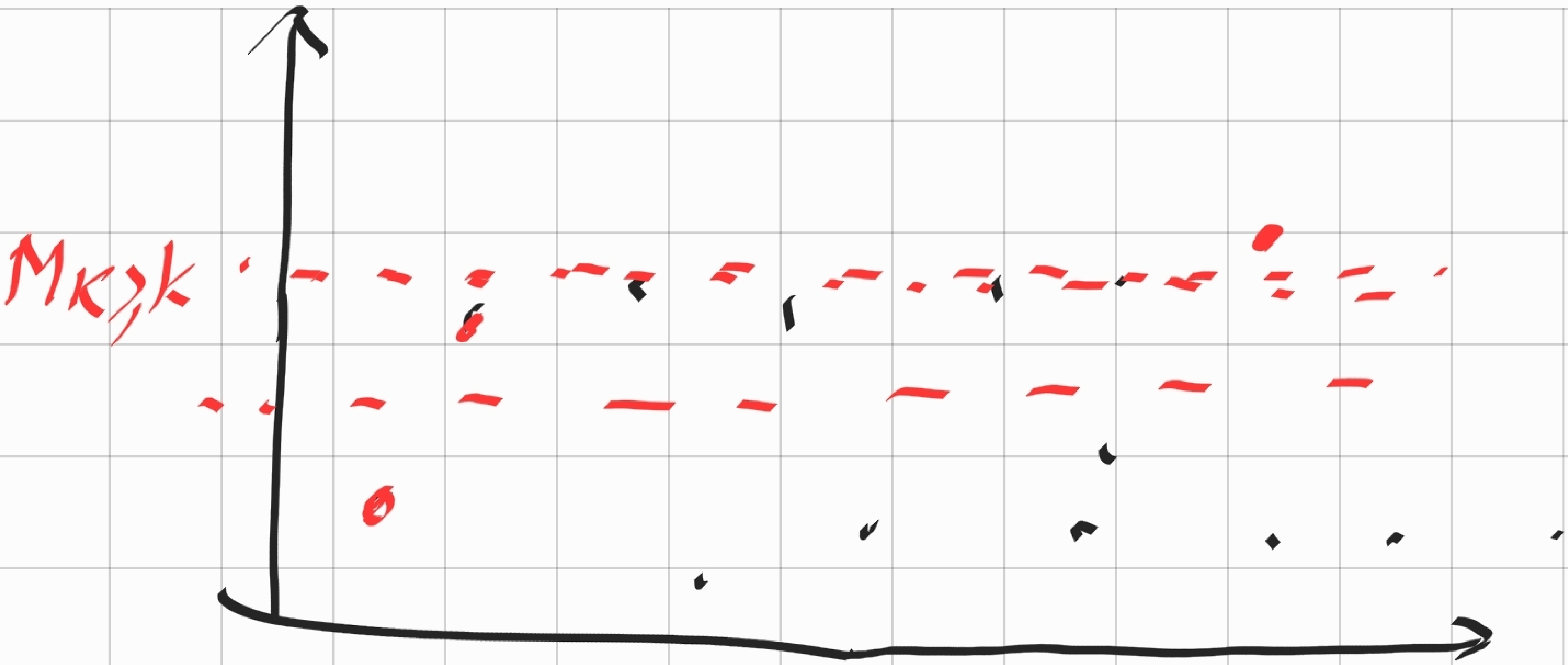
strett. crescente $\Leftarrow$

$a_{n_k} > k$ $\qquad$ $\boxed{a_{n_k} \to +\infty}$

$\sup a_n = +\infty$ $\quad$ $\exists\, n_k$ t.c. n_k strett crescente

$(a_{n_k}$ strett. crescente$)$ $a_{n_k} \to +\infty$

$\Rightarrow \limsup_{n \to +\infty} a_n = +\infty$



$a_{n_k} \geq M_k \geq k \rightarrow +\infty$