ANALISI MATEMATICA B
LEZIONE 53 — 15.2.2021

3. test riepilogo

$$\sum_k \Big(e - \Big(1+\frac{1}{k}\Big)^k\Big)^\alpha \qquad \sum_j \frac{1}{j!}$$

$$\Big(1+\frac{1}{k}\Big)^k \sim \sum_{j=0}^{k} \frac{1}{j!}$$

$$\downarrow \qquad \downarrow$$

$$e \qquad e$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + o(x)$$

$$e - (1+x)^{\frac{1}{x}} = e\Big(1 - e^{\frac{1}{x}\ln(1+x) - 1}\Big)$$

$$e^{\frac{1}{x}\ln(1+x)-1} = e^{-\frac{x}{2}+o(x)} = 1 - \frac{x}{2} + o(x)$$

$$1 - e^{\frac{1}{x}\ln(1+x)-1} = \frac{x}{2} + o(x) \qquad \boxed{e^y = 1+y+o(y)}$$

$$\Big(e - \Big(1+\frac{1}{k}\Big)^k\Big)^\alpha = e^\alpha\Big(\frac{1}{2k} + o\Big(\frac{1}{k}\Big)\Big)^\alpha$$

$$= \frac{e^\alpha}{2^\alpha k^\alpha} + o\Big(\frac{1}{k^\alpha}\Big)$$

$$\sim \frac{e^\alpha}{2^\alpha k^\alpha} \qquad \underline{\alpha > 1}$$

$$f(x) = \begin{cases} (1+x)^{\frac{1}{x}} & \text{se } x \neq 0 \\ e & \text{se } x = 0 \end{cases}$$

è continua? sì

è derivabile?

$$f(x) = e^{\frac{1}{x}\ln(1+x)} = e^{1-\frac{x}{2}+o(x)}$$

$$\boxed{f(x) - e = e\left(x + o(x)\right)}$$

$x \to 0$

$$\frac{f(x)-e}{x} = e + o(1)$$

$$\frac{f(x)-e}{x} - e = o(1)$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = e$$

$$a_k \sim b_k$$

$$e - a_k \overset{?}{\sim} e - b_k$$

$$a_k = e + \frac{1}{k^2} \qquad b_k = e + \frac{1}{k}$$

$$\frac{1}{k^2} \quad \underset{\sim}{\text{No}} \quad \frac{1}{k}$$

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)} = e^{1-\frac{x}{2}+o(x)} = e \cdot e^{-\frac{x}{2}+o(x)}$$

$$\left(e - \left(1+\frac{1}{k}\right)^k\right)^\alpha = \left(e - e \cdot e^{-\frac{1}{2k}+o\left(\frac{1}{k}\right)}\right)^\alpha$$

$$= \left[e\left(1 - e^{-\frac{1}{2k}+o\left(\frac{1}{k}\right)}\right)\right]^\alpha$$

$$= \left[e\left(\cancel{1} - \cancel{1} + \frac{1}{2k} + o\left(\frac{1}{k}\right)\right)\right]^\alpha$$

$$= \frac{e^\alpha}{(2k)^\alpha} + o\left(\frac{1}{k^\alpha}\right)$$

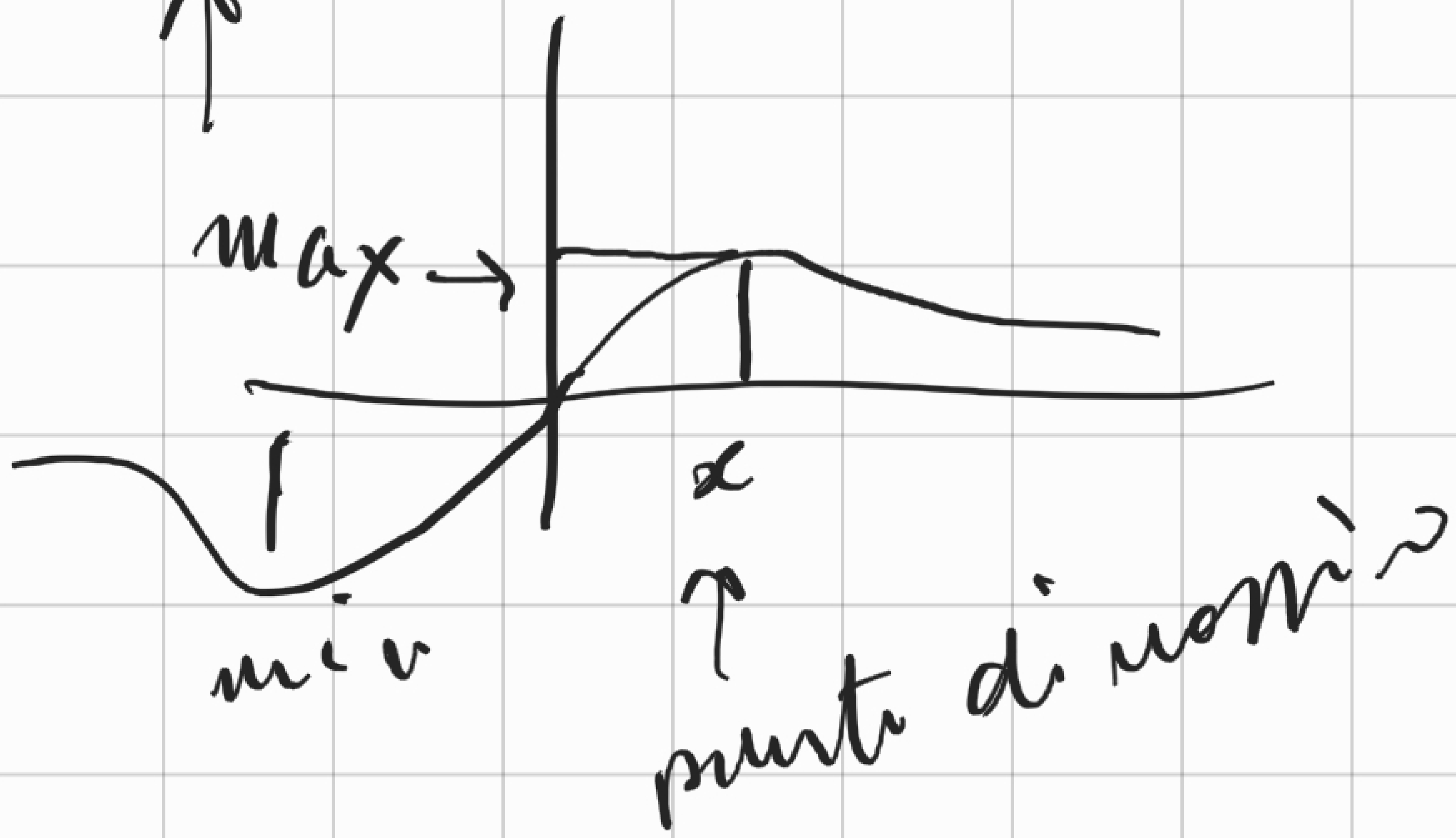
1. $$\inf_{y\in\mathbb{R}} \sup_{x\in\mathbb{R}} \frac{x}{1+x^2+y^2}$$

$$f_y(x) = \frac{x}{1+x^2+y^2}$$

y parametro fissato

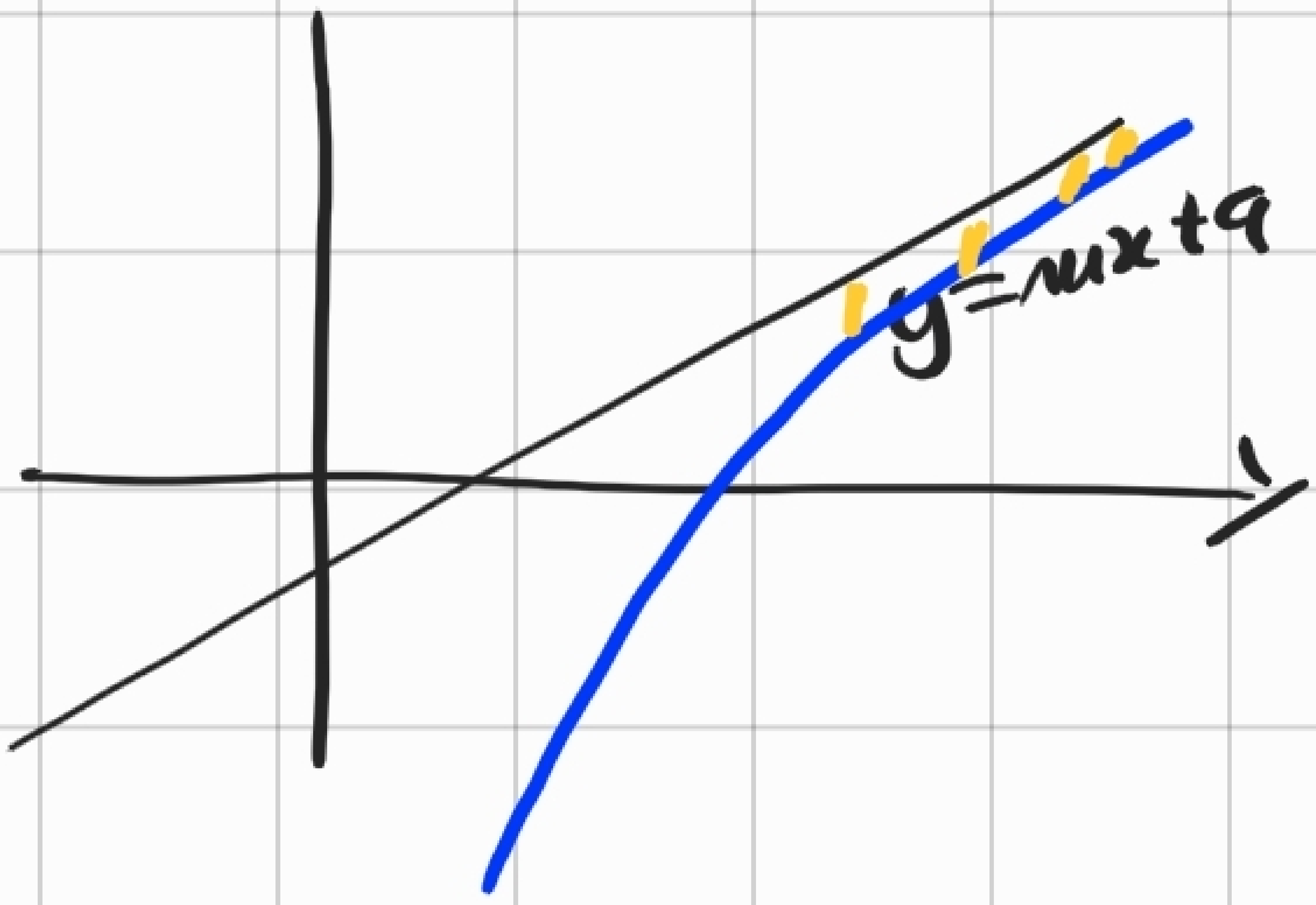
$$g(y) = \max_{x\in\mathbb{R}} f_y(x)$$

□



Nomenclatura

Asintoti



Se $\lim_{\substack{x\to+\infty \\ (-\infty)}} f(x)-(mx+q) = 0$

diremo che $y=mx+q$ è un asintoto per il grafico di f per $x\to+\infty$ $(-\infty)$

se $m\neq 0$ si dice asintoto obliquo

se $m=0$ " " orizzontale

$\boxed{\lim_{x\to+\infty} f(x) = q}$ $\qquad y=q$

Come si trova un eventuale asintoto obliquo?

$$f(x) - (mx + q) \longrightarrow 0 \quad \text{per } x \to +\infty$$

$$\frac{f(x) - mx - q}{x} \longrightarrow 0$$

$$\parallel$$

$$\frac{f(x)}{x} - m - \frac{q}{x} \longrightarrow 0$$

$$\boxed{\frac{f(x)}{x} \to m}$$ per $x \to +\infty$ se c'è asintoto obliquo.

$$\lim_{x \to +\infty} f(x) - mx = q \in \mathbb{R} \Rightarrow y = mx + q$$ è l'asintoto obliquo.

Hopital: $\frac{\infty}{\infty}$

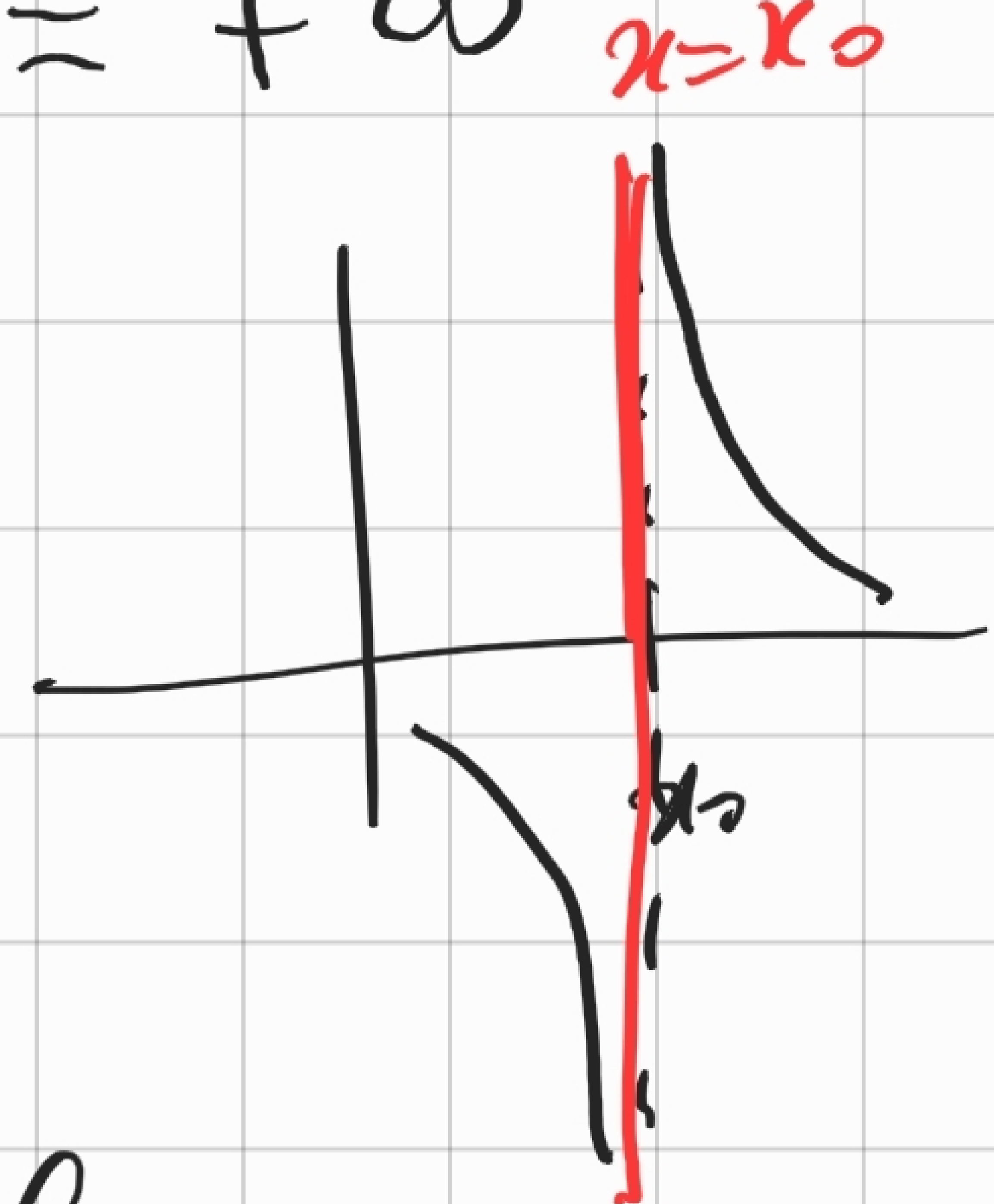
se $f'(x) \to m$

allora $\boxed{\frac{f(x)}{x}} \to m$

per $x \to \infty$

$x_0 \in \mathbb{R}$

Se $\lim\limits_{x \to x_0} |f(x)| = +\infty$



diremo che la retta $x = x_0$

è un <u>asintoto verticale</u>

$x \to +\infty$

$f(x) \sim g(x)$ $\qquad \dfrac{f(x)}{g(x)} \to 1$

$f(x)$ ha asintoto $g(x)$ se $f(x) - g(x) \to 0$

non sono equivalenti.

<u>Esempio</u> $f(x) = x + \ln x$

per $x \to +\infty$

$f(x) \sim x$ $\qquad \dfrac{x + \ln x}{x} \to 1$

ma $f(x) - x = \ln x \to +\infty$

x non è asintoto obliquo

<u>Esempio</u> $f(x) = \dfrac{1}{x}$ $\qquad g(x) = \dfrac{1}{x^2}$

$f(x) - g(x) \to 0$ per $x \to +\infty$

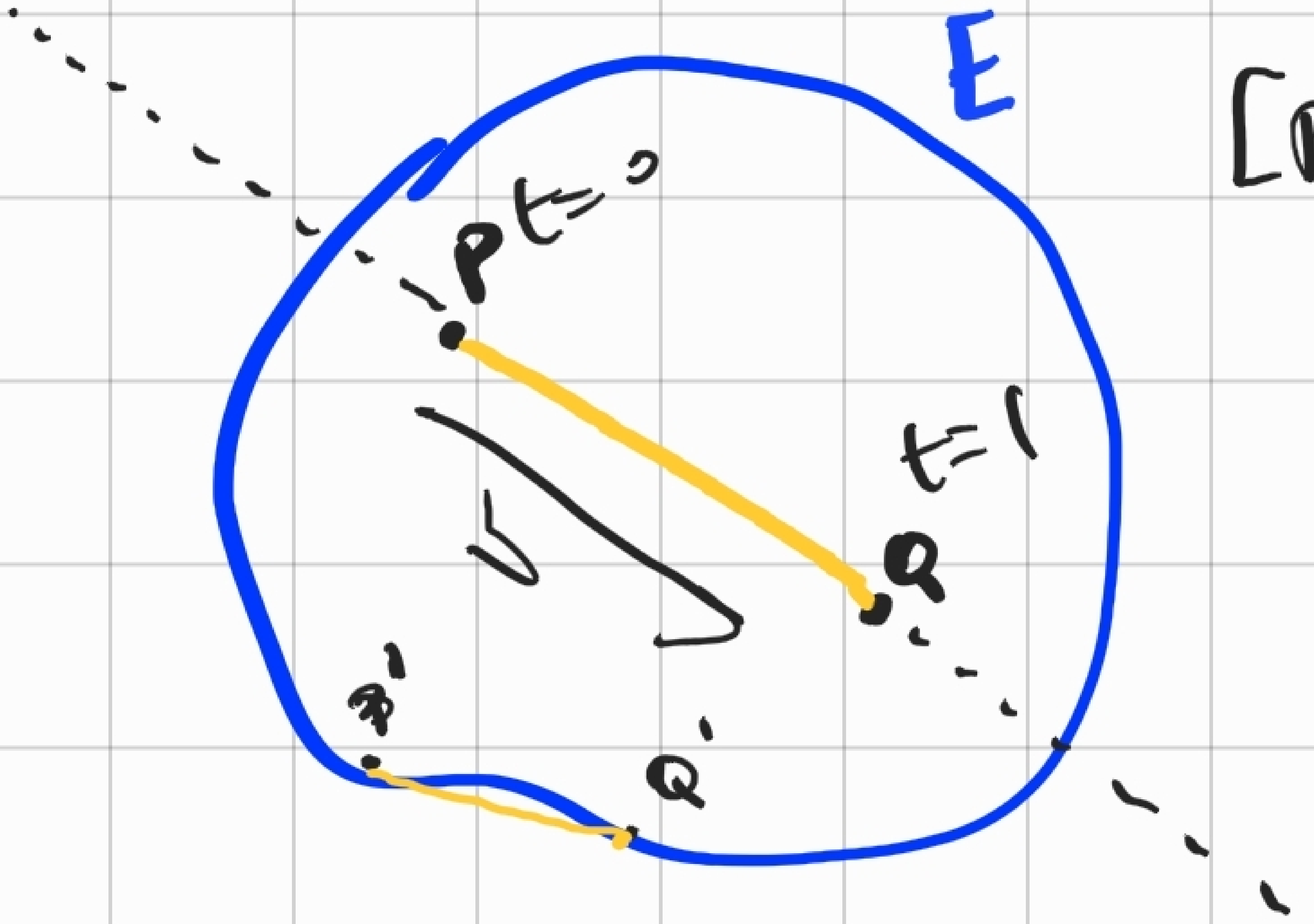
ma $\dfrac{f(x)}{g(x)} = \dfrac{1}{x} \cdot \dfrac{x^2}{1} \to +\infty$

f e g hanno lo stesso asintoto orizzontale

$y = 0$

ma non sono asintoticamente equivalenti.

CONVESSITA'



$$[P,Q] = \{(1-t)P + tQ : t\in[0,1]\}$$

$t\in\mathbb{R}$

$$P + tv = P + t(Q-P) = (1-t)P + tQ$$

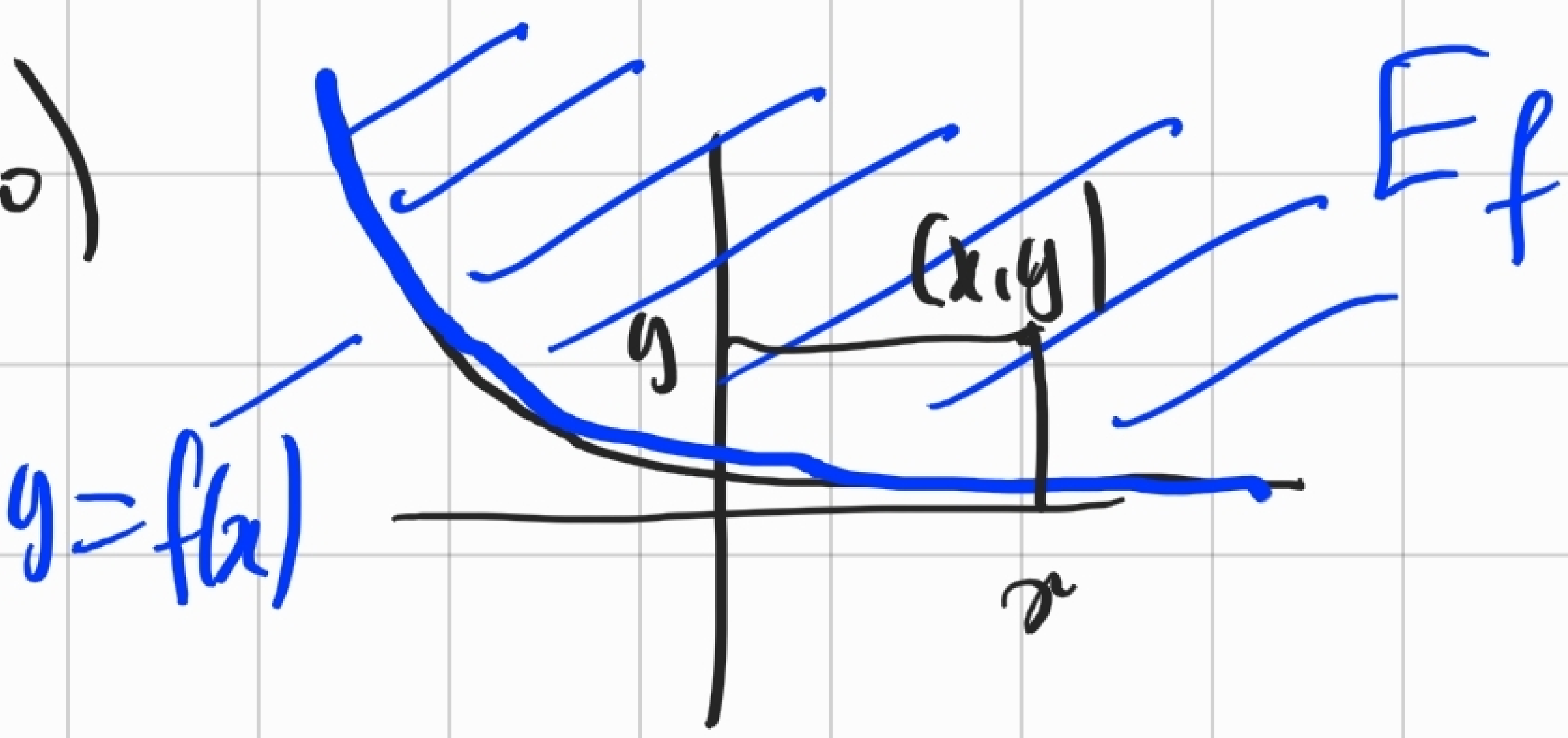
Def $E\subseteq\mathbb{R}^n$ si dice essere convesso se $\forall\, P,Q\in E$ il segmento $[P,Q] = \overline{PQ} \subseteq E$.

Def $f : A\subseteq\mathbb{R}\to\mathbb{R}$ si dice essere convessa se

E_f convesso

$$E_f = \{(x,y)\in\mathbb{R}^2 : y \geq f(x)\}$$

epigrafico (sopragrafico) è convesso.



Equivalentemente:

f convessa

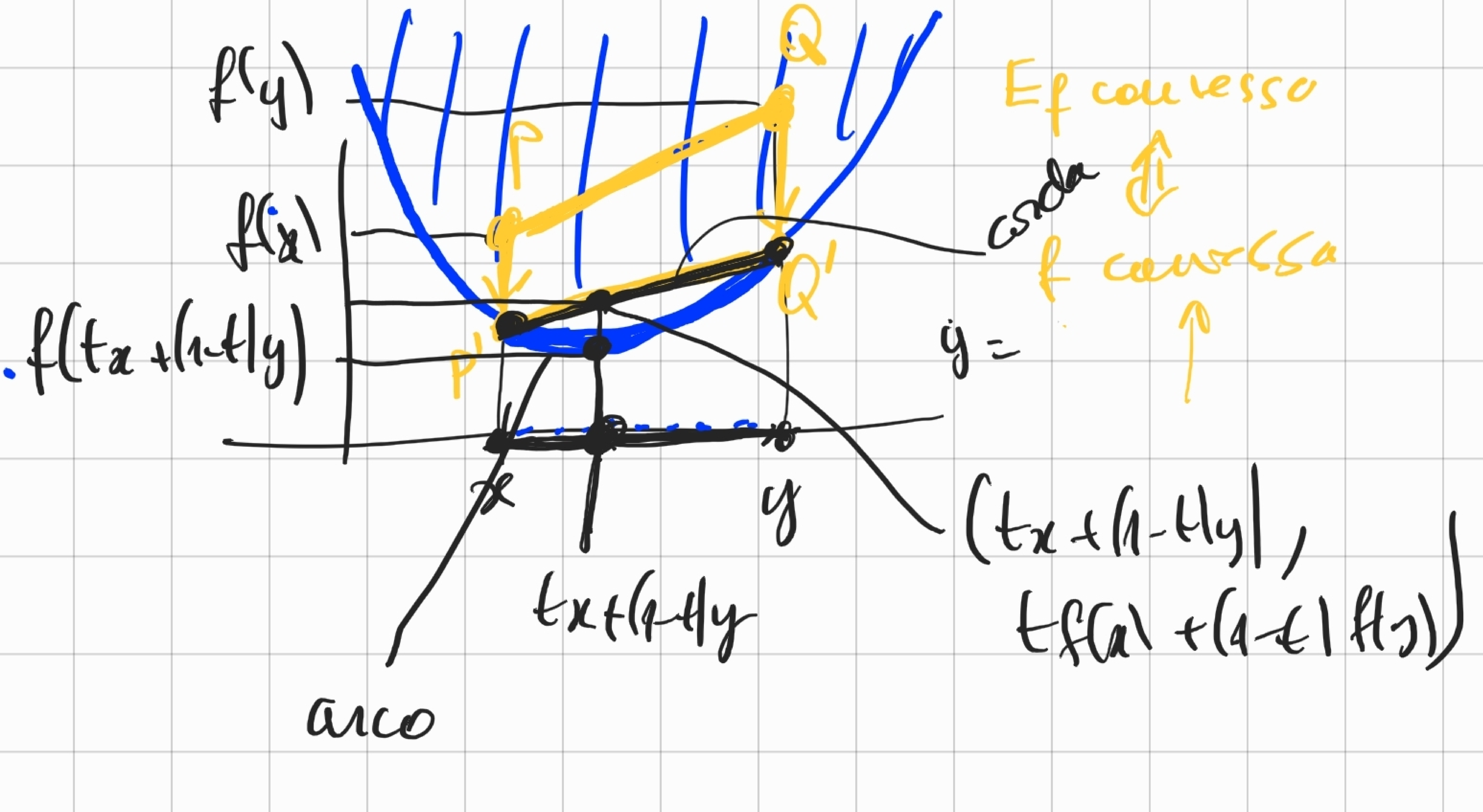
<u>def</u> (equivalente) $f: A \subseteq \mathbb{R} \to \mathbb{R}$

è convessa se
1) A è un intervallo
2) $\forall x, y \in A \ \forall t \in [0,1]$

$$\boxed{f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)}$$

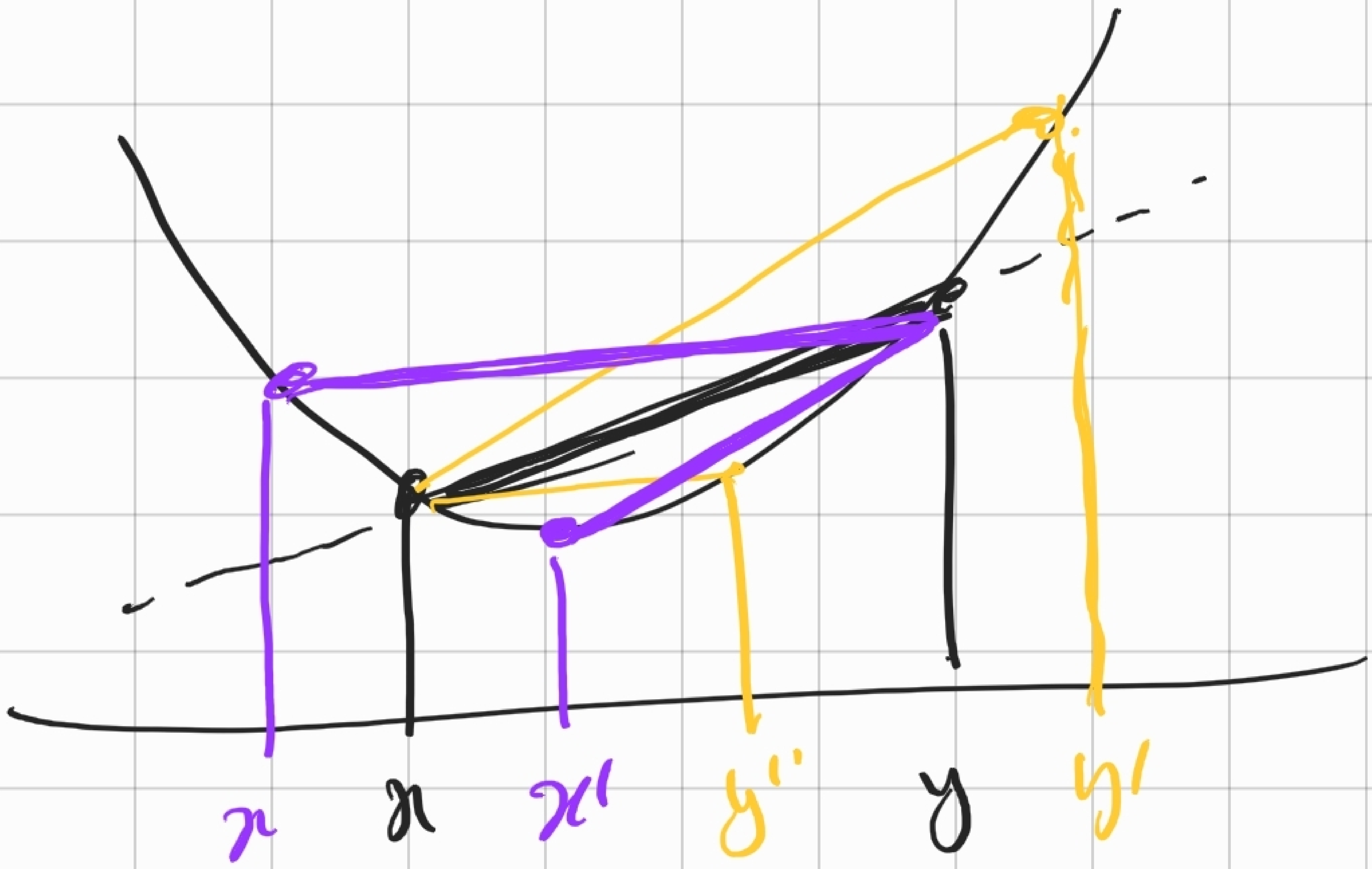
($\geq$) concava

<u>Oss</u> $A \subseteq \mathbb{R}$ è convesso $\Leftrightarrow$ A intervallo



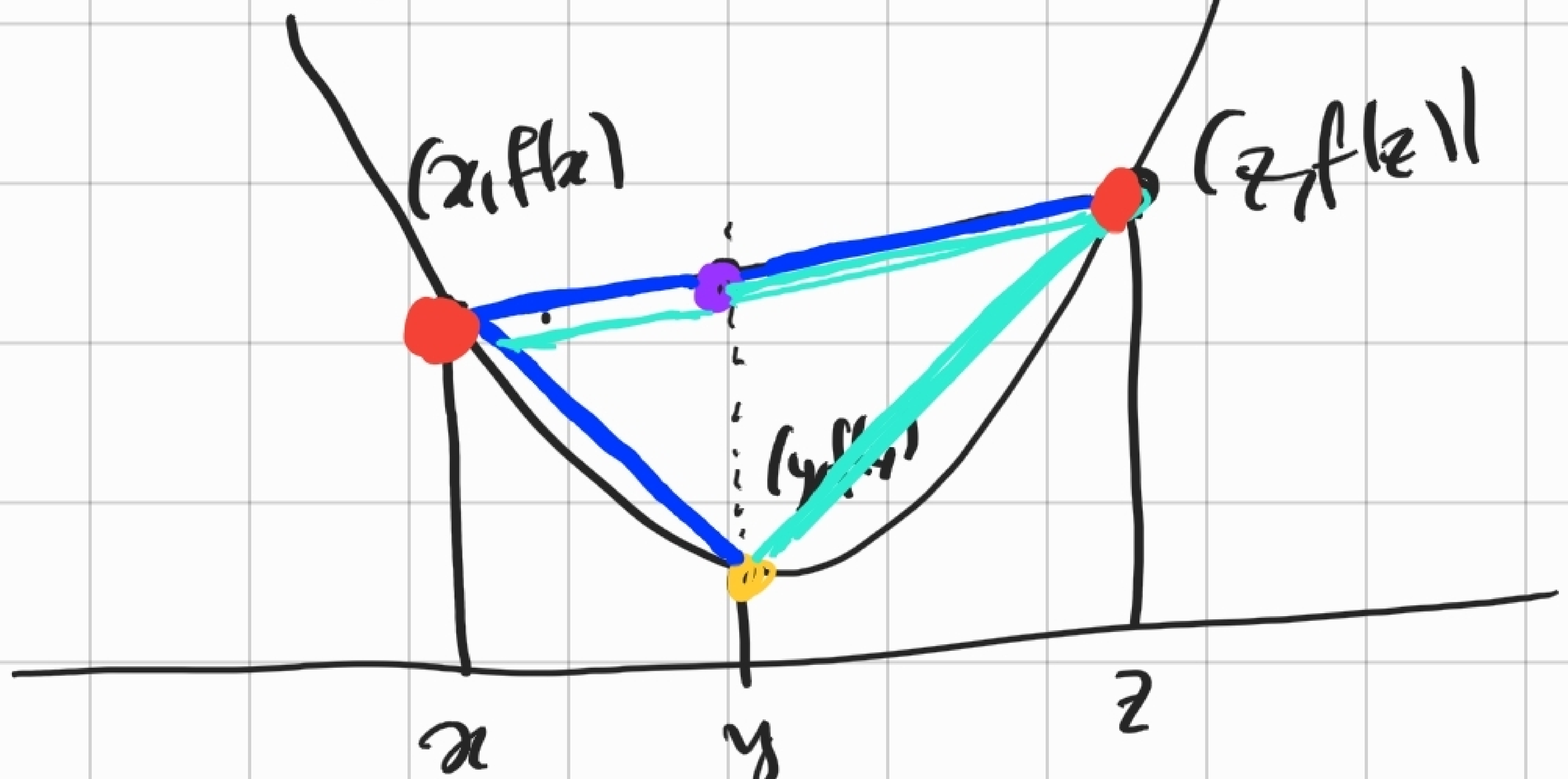
Rapporti incrementali data f

$$R(x,y) = \frac{f(y)-f(x)}{y-x}$$ è la pendenza della corda



f convessa se e solo se:

$R(x,y) = R(y,x)$ è crescente sia in x che in y.

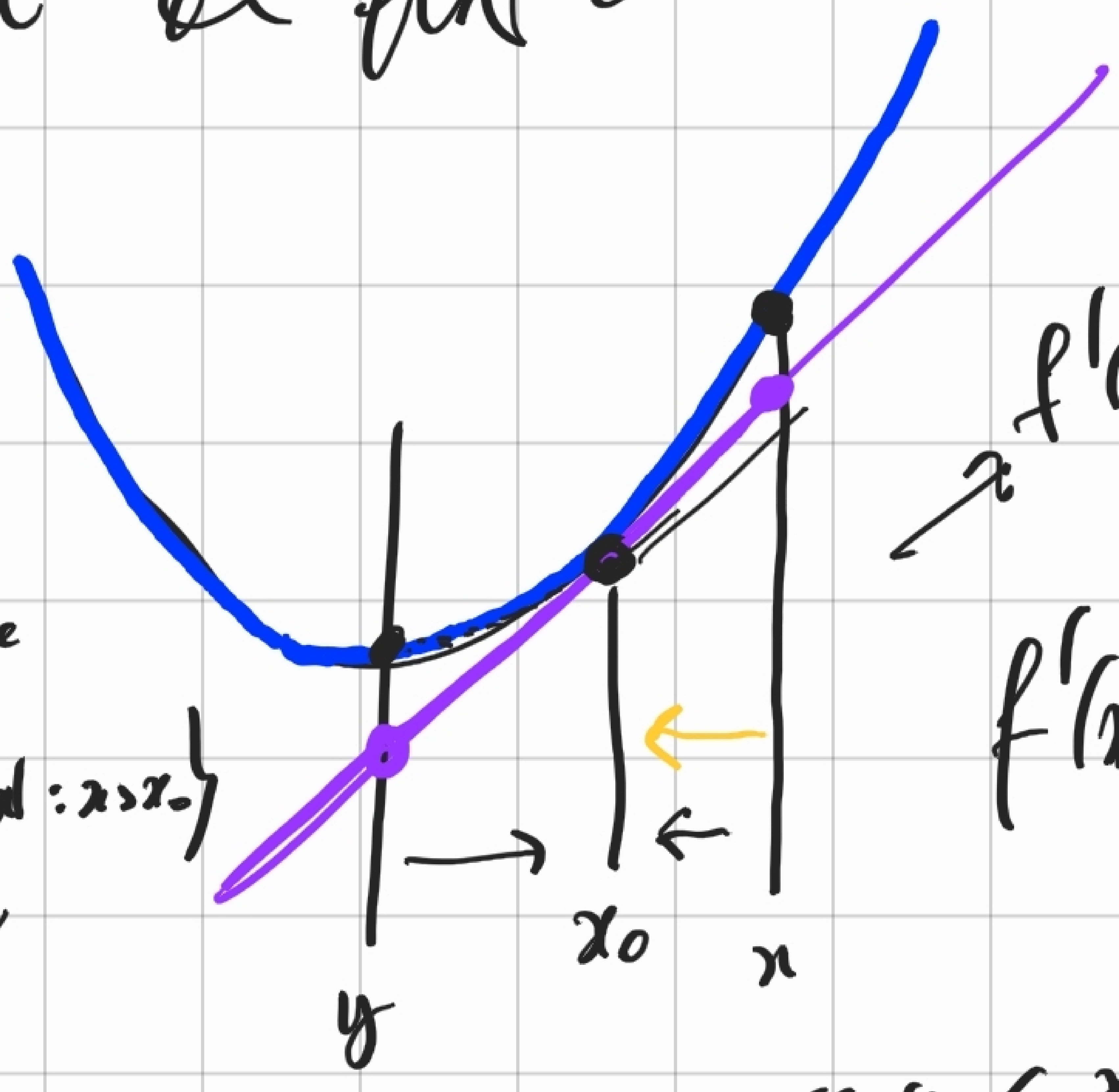


$x < y < z$

$$\begin{cases} \text{stà sopra} \\ R(x,z) \geq R(x,y) \\ R(y,z) \geq R(x,z) \end{cases} \Rightarrow R(x,y) \leq R(y,z)$$

Teorema Se f è derivabile, $f: I \to \mathbb{R}$:

f è convessa
$\Updownarrow$
il grafico di f "sta sempre sopra" la retta tangente al grafico



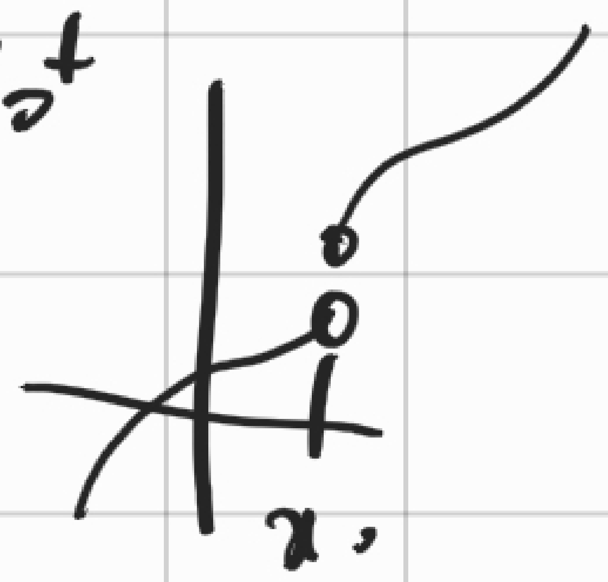
$$f'(x_0) = \lim_{x \to x_0^+} R(x_0, x)$$

$f'(x_0) \leq R(x_0, x)$

$\forall x > x_0$

Se $g(x)$ crescente

$$\lim_{x \to x_0^+} g(x) = \inf \{g(x) : x > x_0\}$$



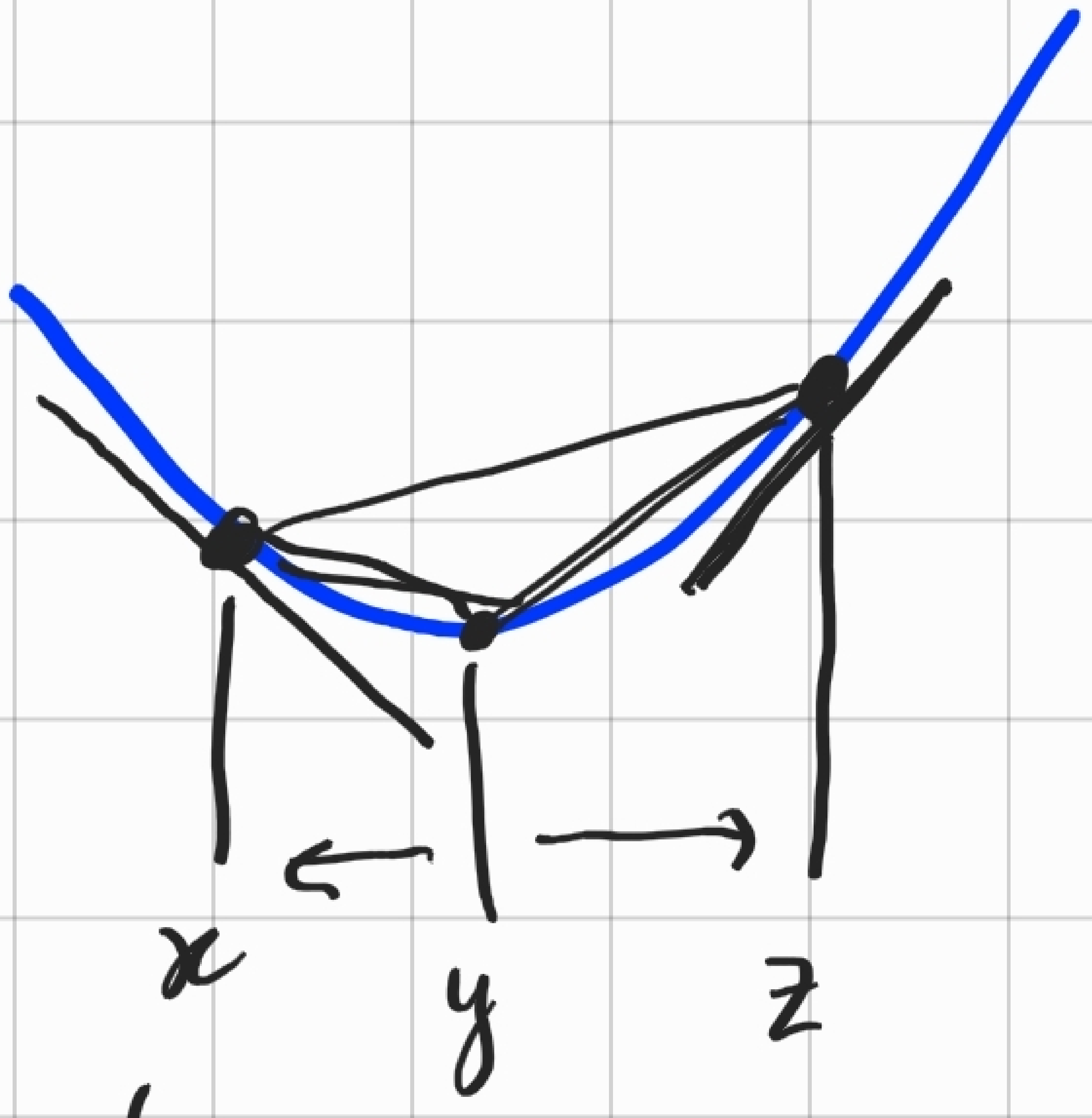
$\forall y < x_0$

$R(x_0, y) = R(y, x_0)$

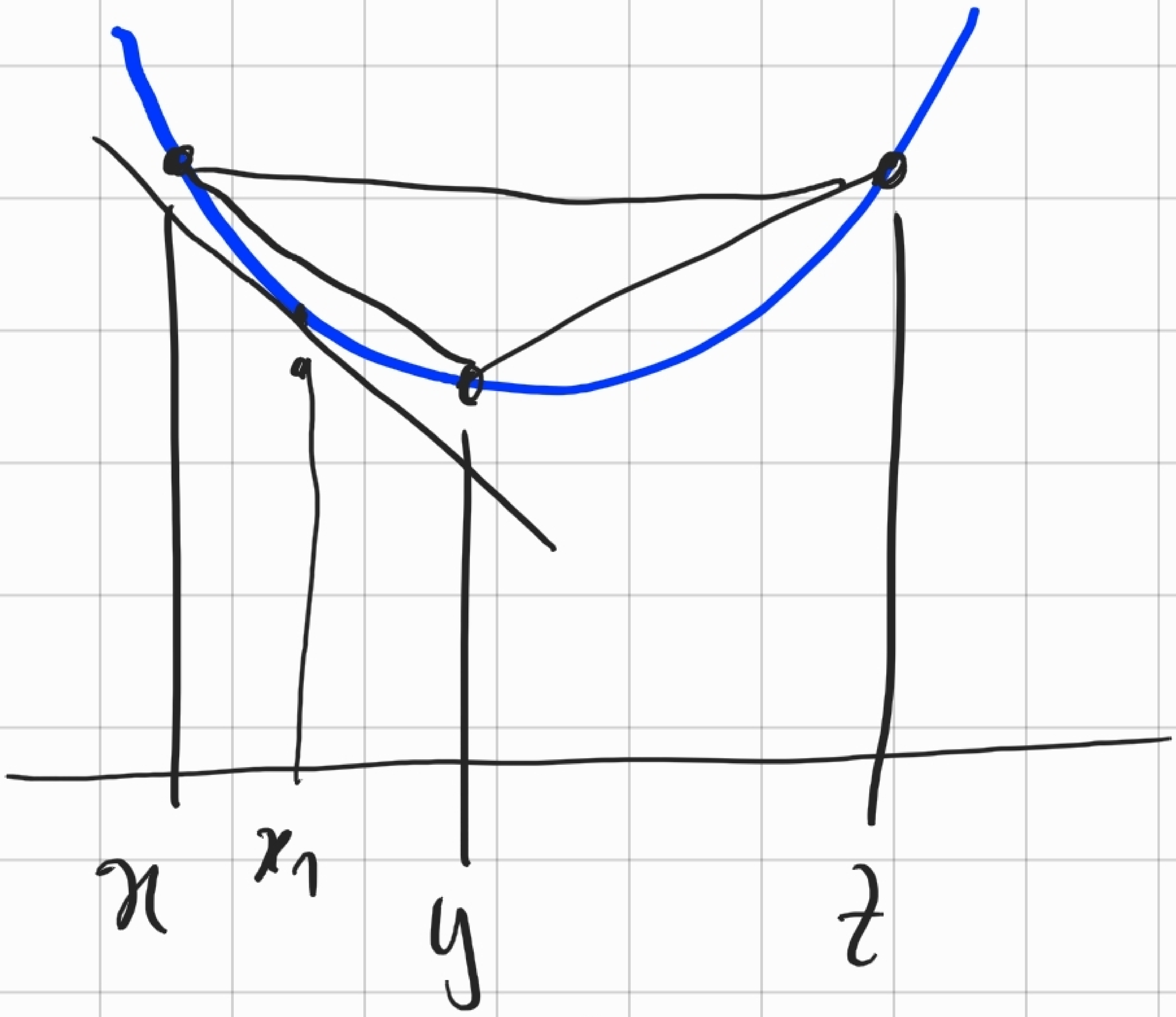
f convessa

$\Updownarrow$

f' crescente



$$f'(x) \leq R(x,y) \leq f'(z)$$



$$\begin{array}{ccc} R(x,y) & \overset{?}{\leq} & R(y,z) \\ \| & \Uparrow & \| \\ f'(x_1) & \leq & f'(x_2) \end{array}$$

$$x < x_1 < y < x_2 < z$$

$\square$

Se f derivabile 2 volte

f' crescente $\iff f'' \geq 0$.