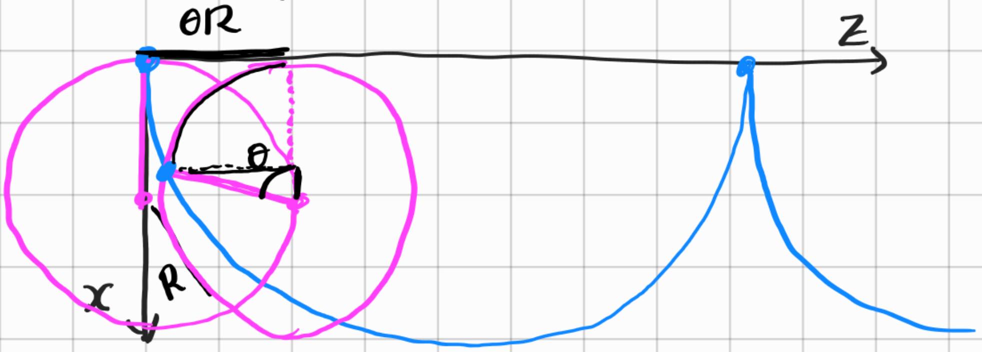
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 2 - 3.2.2023

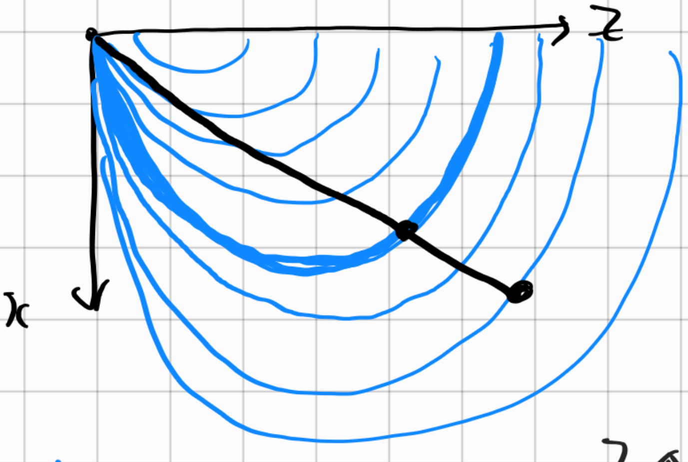
Brachistocrona (continua)

$$u(x) = R \arccos\left(\frac{R-x}{R}\right) - \sqrt{R^2-(x-R)^2}$$

Il grafico di u è una cicloide

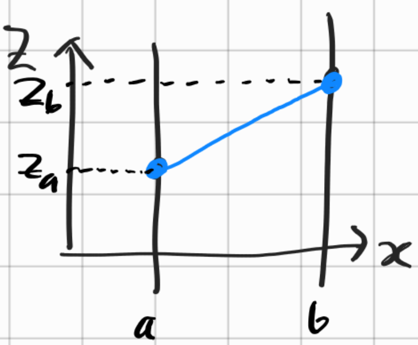


$$\begin{cases} x = R - R\cos\theta \\ z = \theta R - R\sin\theta \end{cases} \qquad \begin{cases} \cos\theta = \dfrac{R-x}{R} \\ z = R \arccos \dfrac{R-x}{R} - R\sqrt{1-\left(\dfrac{R-x}{R}\right)^2} \end{cases}$$



Altri esempi

Geodetiche:



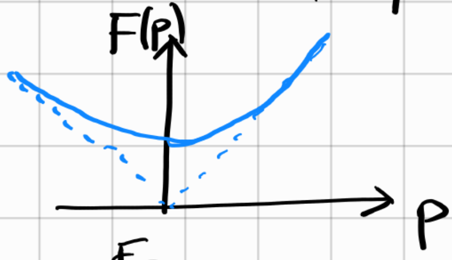
$$\begin{cases} \mathcal{F}(u) = \displaystyle\int_a^b \sqrt{1+(u'(x))^2}\,dx \longrightarrow \min \\ u(a) = z_a \\ u(b) = z_b \end{cases}$$

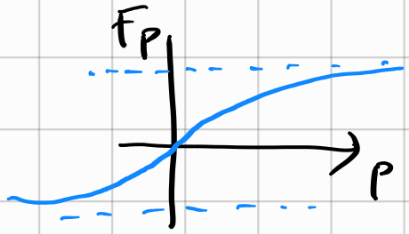
$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\,dx$$

$$F(x, z, p) = \sqrt{1+p^2}, \quad F_z = 0, \quad F_p = \frac{p}{\sqrt{1+p^2}}$$

E.L. $\quad F_z = \frac{d}{dx} F_p$

$$0 = \frac{d}{dx}\left[\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right]$$

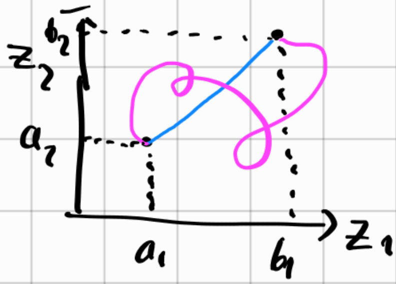




$$F_p(u'(x)) = \frac{u'(x)}{\sqrt{1+(u'(x))^2}} = \text{cost}$$

$u'(x) = m \qquad u(x) = mx + q.$

In forma parametrica: $\quad \underline{z} = u(x) \qquad \underline{z} = (z_1, z_2)$



$\underline{u} : [0,1] \to \mathbb{R}^2$

$\underline{u}(0) = \underline{a} = (a_1, a_2)$

$\underline{u}(1) = \underline{b} = (b_1, b_2)$

$$\mathcal{F}(\underline{u}) = \int_0^1 |\underline{u}'(x)|\,dx \qquad F(x, \underline{z}, \underline{p}) = |\underline{p}|$$

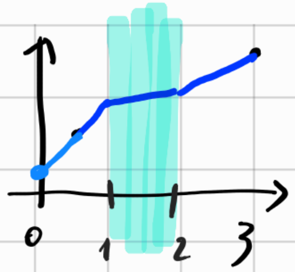
$$F_{\underline{p}} = \frac{\underline{p}}{|\underline{p}|} \qquad F_{\underline{z}} = 0$$

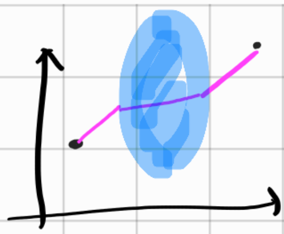
E.L. $\quad \frac{d}{dx} \frac{\underline{u}'(x)}{|\underline{u}'(x)|} = 0$

$\frac{\underline{u}'(x)}{|\underline{u}'(x)|} = \text{cost.} \qquad \underline{u}(x)$ ha direzione costante.

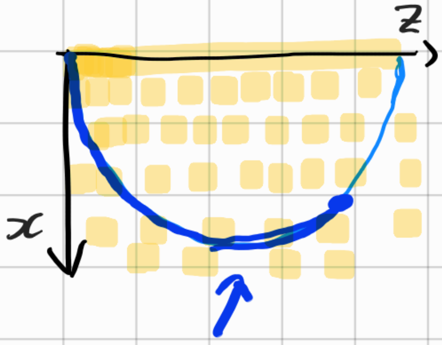
$\Rightarrow$ Im $\underline{u}$ è contenuto in una retta.

Diffrazione





$$\mathcal{F}(u) = \int_0^3 \sqrt{1+(u'(x))^2}\cdot g(x)\,dx \quad \text{oppure} \quad \mathcal{F}(u) = \int_0^3 \sqrt{1+(u')^2}\cdot g(x,z)\,dx$$



anche la brachistocrona rientra in questa forma

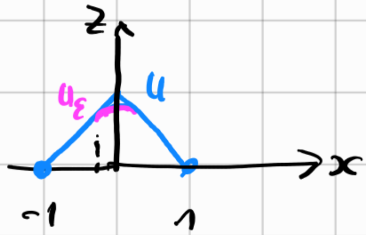
$$\boxed{\mathcal{F}(u) = \int \frac{\sqrt{1+(u'(x))^2}}{\sqrt{x}}\,dx}$$

Controesempi

(1) $$\mathcal{F}(u) = \int_{-1}^{1} \left(1-(u'(x))^2\right)^2 dx = \int_{-1}^{1} F(x,u(x),u'(x))\,dx$$

$$F(x,z,p) = (1-p^2)^2$$

$$\begin{cases} \mathcal{F}(u) \to \min \\ u(-1) = 0 \\ u(1) = 0 \end{cases}$$



$u_0(x) = 1-|x|$

$u_0 \notin C^1$

ma è derivabile quasi ovunque. ($x \neq 0$)

$F(x,u(x),u'(x)) = 0$

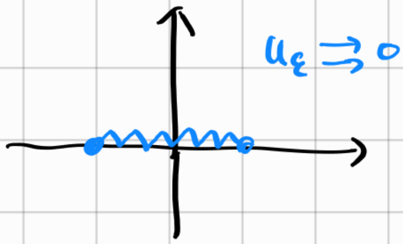
$\exists\, u_\varepsilon \in C^1$ tali che $u_\varepsilon \to u_0$ per $\varepsilon \to 0$

$\mathcal{F}(u_\varepsilon) \to 0$

$\mathcal{F}$ non ha minimo in C^1

(2) $$\mathcal{F}(u) = \int_{-1}^{1} \left[\left(1-(u')^2\right)^2 + u^2(x)\right] dx$$

$$F(x,z,p) = (1-p^2)^2 + z^2$$



$\mathcal{F}(u_\varepsilon) \to 0$ per $\varepsilon \to 0$

$\mathcal{F}(u) = 0 \Rightarrow \int u^2 = 0 \Rightarrow u \equiv 0$ q.o.

$\Downarrow$

$u' = 0$

$\int (1-u'^2)^2 \Leftarrow$

$= \int 1 > 0.$

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(3)

$$\begin{cases} \mathcal{F}(u) = \int_0^1 (u'(x))^3 \, dx \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

$F(x,z,p) = p^3 \qquad F_z = 0$

$F_p = 3p^2$

E.L.:

$0 = \frac{d}{dx} 3(u')^2$

$(u')^2 = \text{cost.} \qquad u' = m$

$u(x) = mx + q$

$u(0) = u(1) = 0$

$\Rightarrow u(x) \equiv 0.$

ma $u(x) \equiv 0$ non è minimo.

OSS: $u(0) = u(1) = 0$

$\int_0^1 u' = [u]_0^1 = 0.$

$$u_k(x) = \begin{cases} k \cdot x & \text{per } x \in [0, \frac{1}{k^2}] \\ \frac{1}{k} & \text{per } x \in [\frac{1}{k^2}, 1 - \frac{1}{2k^2}] \\ 2k(1-x) & \text{per } x \in [\underbrace{1 - \frac{1}{2k^2}, 1}] \end{cases}$$

$$\mathcal{F}(u_k) = \int_{-1}^1 (u_k')^3 = \int_0^{1/k^2} (k)^3 + \int_{1-\frac{1}{2k^2}}^1 (-2k)^3$$

$$= \frac{k^3}{k^2} - \frac{8k^3}{2k^2} = -3k \to -\infty.$$

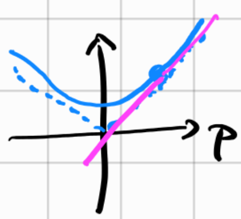
Le cose possono andare male se $p \mapsto F(x,z,p)$ non è convessa.

Come uso la convessità?

Brachistocrona

$$\mathcal{F}(u) = \int_a^b \frac{\sqrt{1+u'^2}}{\sqrt{x}}\,dx$$

$$F(x,p) = \frac{\sqrt{1+p^2}}{\sqrt{x}}$$



(*) $\forall x:$ $\quad \underline{F(x,p) - F(x,p_0)} \geq F_p(x,p_0)\,(p-p_0)$

Data u_0 candidata minimo vorrei mostrare che $\mathcal{F}(u) \geq \mathcal{F}(u_0)$ $\forall u$.

$$\mathcal{F}(u) - \mathcal{F}(u_0) = \int_a^b \left[\underline{F(x,u'(x)) - F(x,u_0'(x))}\right] dx$$

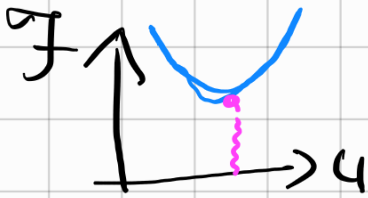
$\begin{cases} p = u'(x) \\ p_0 = u_0'(x) \end{cases}$ in (*) $\Rightarrow$ $\displaystyle \geq \int_a^b \underbrace{F_p(x,u_0'(x))}_{\text{è costante se } u_0 \text{ soddisfa E.L.}} \cdot (u'(x) - u_0'(x))\,dx$

$$= C \int_a^b (u(x) - u_0(x))'\,dx = C\left[\,u - u_0\,\right]_a^b = 0$$

$\begin{cases} u(b) = u_0(b) \\ u(a) = u_0(a) \end{cases}$

u_0 è veramente un minimo.

In effetti se $\forall x$ $\quad (z,p) \longmapsto F(x,z,p)$ è convessa



allora $\mathcal{F}$ è convesso:

$$\mathcal{F}(\underbrace{(1-t)u + tv}) \overset{?}{\leq} (1-t)\,\mathcal{F}(u) + t\,\mathcal{F}(v)$$

$$\parallel$$

$$\int F(x, \underbrace{(1-t)u + tv, (1-t)u' + tv'})\,dx$$

$$\underset{\uparrow}{\leq} \int \left[(1-t)\,F(x,u,u') + t\,F(x,v,v')\right] dx$$

$\forall x$: F convessa in (z,p)

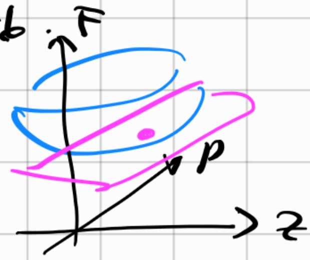
Fatto generale Se $\mathcal{F}$ è convesso e $\delta\mathcal{F}(u_0,\varphi) = \frac{\partial \mathcal{F}}{\partial \varphi}(u_0) = 0 \quad \forall \varphi$.

allora u_0 è minimo di $\mathcal{F}$.

Teo Se $\forall x \quad (z,p) \longmapsto F(x,z,p)$ è convesso

e se u_0 soddisfa E.L: $\left[\, F_z(x,u_0,u_0') = \frac{d}{dx} F_p(x,u_0,u_0') \,\right]$

Allora u_0 è minimo di
$$\begin{cases} \mathcal{F}(u) = \int_a^b F(x,u,u')\,dx \\ u(a) = z_a \\ u(b) = z_b \end{cases}$$



dim

$$\mathcal{F}(u) - \mathcal{F}(u_0) = \int \left[F(x,u,u') - F(x,u_0,u_0') \right] \geq \#$$

Uso convessità: $\underbrace{F(x,z,p) - F(x,z_0,p_0)}$ $\geq F_z(x,z_0,p_0)\cdot(z-z_0) + F_p(x,z_0,p_0)\cdot(p-p_0)$

$$\# \geq \int \left[F_z(x,u_0,u_0')(u-u_0) + F_p(x,u_0,u_0')(u'-u_0') \right] dx$$

$$\overset{\text{Per parti}}{=} \int \underbrace{\left[F_z - \frac{d}{dx} F_p \right]_{\substack{z=u_0 \\ p=u_0'}}}_{E.L.=0} \cdot (u-u_0)\, dx + \cancel{\left[F_p \cdot (u-u_0) \right]_a^b}$$

$(u-u_0)(a) = (u-u_0)(b) = 0$

$$= 0. \qquad \square$$