ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 6 - 16.3.2023

Esempio (eq. di Newton)

$z = u(x)$
$p = u'(x)$

$$F(x,z,p) = \frac{1}{2} m p^2 - V(z)$$

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\, dx \quad = \text{azione}$$

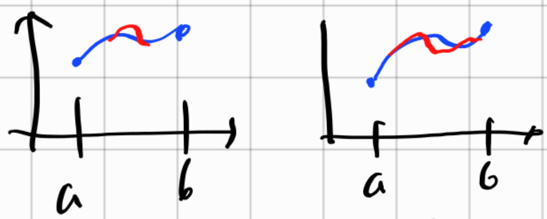
$$F_z = -\nabla V \qquad F_p = m p$$

E.L.: $$\frac{d}{dx} m u'(x) = -\nabla V(u(x))$$

$$m u''(x) = -\nabla V(u(x))$$

$$m a = F$$

VARIAZIONI INTERNE



VAR. EST. $\quad$ VAR. INT.

$u_\varepsilon(x) = u(x) + \varepsilon \varphi(x)$

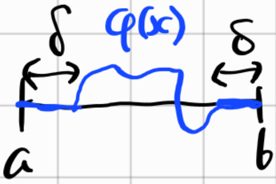
$\varphi \in C_c^\infty([a,b])$

$$\eta_\varepsilon(y) = y + \boxed{\varepsilon \cdot \varphi(y)}$$

$$\underbrace{u_\varepsilon(x)} = u(\underbrace{\eta_\varepsilon^{-1}(x)}_{y})$$

mi serve: $\eta_\varepsilon : [a,b] \to [a,b]$ sia biiettiva.

se ε abbastanza piccolo $\varepsilon \cdot \|\varphi\|_{L^\infty} < \delta$



quindi $\eta_\varepsilon([a,b]) \subseteq [a,b]$

Vorrei anche $\eta_\varepsilon' > 0$ $\quad \eta_\varepsilon' = 1 + \varepsilon\varphi'$

se ε abbastanza piccolo $\varepsilon \|\varphi'\|_{L^\infty} < 1$ $\quad \eta_\varepsilon' > 0$.

u_ε è ben definita.

Se $\mathcal{J}(u) \leq \mathcal{J}(u_\varepsilon)$ $\quad$ (u minimo)

Allora:

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}(u_\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_a^b F(x, u_\varepsilon(x), u_\varepsilon'(x))\,dx =$$

$$\begin{aligned} x &= \eta_\varepsilon(y) \\ dx &= \eta_\varepsilon'(y)\,dy \end{aligned}$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_a^b F\left(\eta_\varepsilon(y), u(y), \frac{u'(y)}{\eta_\varepsilon'(y)}\right)\eta_\varepsilon'(y)\,dy$$

$$u_\varepsilon'(x) = u'(y) \cdot \frac{1}{\eta_\varepsilon'(y)}$$

$$= \int_a^b \left\{\left[F_x \cdot \varphi - F_p \cdot \frac{u'(y)\,\varphi'}{(\eta_\varepsilon')^2}\right]\cdot\eta_\varepsilon' + F\varphi'\right\}_{\varepsilon=0} dy$$

$$\boxed{\begin{aligned} u_\varepsilon(x) &= u(\eta_\varepsilon^{-1}(x)) \\ \eta_\varepsilon(y) &= y + \varepsilon\varphi(y) \\ \eta_\varepsilon'(y) &= 1 + \varepsilon\varphi'(y) \end{aligned}}$$

$$= \int_a^b \left[F_x \cdot \varphi - F_p u' \varphi' + F\varphi'\right] dy$$

per parti:

$$= \int_a^b \left[-\int F_x - F_p u' + F\right]\varphi'\,dy$$

<u>Lemma fondamentale</u>: (I) $\int f \cdot \varphi = 0 \quad \forall \varphi \in C_c^\infty$

se $f \in C^0 \Rightarrow f \equiv 0$

(II) $\int f \cdot \varphi' = 0 \quad \forall \varphi \in C_c^\infty \quad f \in C^1$

$$\left(\quad 0 = \int f \cdot \varphi' = - \int f' \cdot \varphi \quad \Rightarrow \quad f' \equiv 0 \quad \Rightarrow \quad f = c. \quad \right)$$

Deduco: $\tau(x) = u' \cdot F_p - F + \int F_x$ è costante.

$$= u'(x) \cdot F_p(x, u(x), u'(x)) - F(x, u(x), u'(x)) + \int_a^x F_x(t, u(t), u'(t))\,dt$$

Abbiamo visto che se F è abbastanza regolare (C^1) e se u è minimo <u>locale</u> in C^0 di $\mathcal{F}(u) = \int F(x, u, u')$

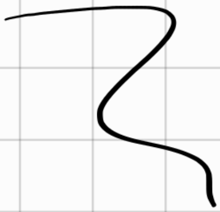
<u>allora</u> $\tau(x)$ è costante.

Nel esempio precedente (Newton)

$$F(x, z, p) = \frac{1}{2} m p^2 - V(z) \qquad F_p = m \cdot p, \quad F_x = 0$$

$$\tau(x) = u' \cdot m \cdot u' - \left[\frac{1}{2} m (u')^2 - V(u) \right] + 0$$

$$= \underbrace{\frac{1}{2} m (u')^2}_{\text{E. cinetica}} + \underbrace{V(u)}_{\text{E. potenziale}} = \underbrace{\text{costante}}_{\text{conservazione energia}}$$



Esempi di funzioni di più variabili

$u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\mathcal{F}(u) = \int_\Omega F(x, u(x), \nabla u(x))\, dx$$

Se u è minimo allora $\forall \varphi \in C_c^\infty(\Omega)$

$$0 = \delta\mathcal{F}(u,\varphi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}(u+\varepsilon\varphi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega F(x, u+\varepsilon\varphi, \nabla(u+\varepsilon\varphi))\, dx$$

$$= \int_\Omega \left[F_z(x,u,\nabla u)\cdot\varphi + F_p(x,u,\nabla u)\cdot\nabla\varphi\right] dx$$

prodotto scalare

$$= \int_\Omega \left[F_z(x,u,\nabla u) - \mathrm{div}_x\left(F_p(x,u,\nabla u)\right)\right]\varphi\, dx + \int_{\partial\Omega} \varphi\, F_p(\ldots)\cdot\nu_\Omega$$

teorema della divergenza

$$\left[\begin{array}{c} \displaystyle\int_\Omega \mathrm{div}(\varphi F_p) = \int_{\partial\Omega} \varphi\, F_p\cdot\nu_\Omega(x)\, d\sigma(x) \\ \| \\ \displaystyle\int_\Omega \left[\nabla\varphi\cdot F_p + \varphi\cdot\mathrm{div}(F_p)\right] \end{array}\right]$$

- Se u ha dato al bordo assegnato $u \equiv g$ su $\partial\Omega$
  $\varphi \equiv 0$ su $\partial\Omega$

Vale EL : $\quad F_z(x,u,\nabla u) = \mathrm{div}_x \cdot F_p(x,u,\nabla u)$.

- Se u è libera al bordo vale comunque EL:

$$\underbrace{F_p(x,u(x),\nabla u(x))\cdot\nu_\Omega(x)}\; = 0 \quad \forall x \in \partial\Omega.$$

Esempi (Dirichlet)

$$\mathcal{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega f(x)\cdot u$$

$$F(x,z,p) = \frac{1}{2}|p|^2 + f(x)\cdot z$$

$$F_z = f(x) \quad , \quad F_p = p$$

E.L: $\quad \mathrm{div}_x \nabla u = f(x)$

Se il dato al bordo è assegnato $\quad \begin{cases} \Delta u = f & \text{su } \Omega \\ u = g & \text{su } \partial\Omega. \end{cases}$ ← Dirichlet

Se i dato al bordo è libero $\quad \begin{cases} \Delta u = f & \text{su } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{su } \partial\Omega \end{cases}$ ← Neumann.

$F_p \cdot \nu_\Omega = 0$

$\nabla u \cdot \nu_\Omega = 0$

Esempio (superfici minime) $\quad u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\mathcal{J}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx + \int_\Omega H(x)\cdot u(x)\, dx$$

$$F(x,z,p) = \sqrt{1+|p|^2} + H(x)\cdot z$$

$$F_z = H(x) \quad , \quad F_p = \frac{p}{\sqrt{1+|p|^2}}$$

E.L: $\quad \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x).$

↑ curvatura del profilo di u.

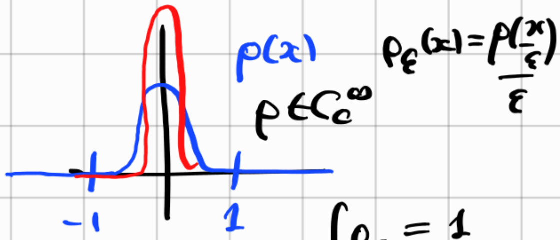
Vorremo estendere il Lemma fondamentale:

Lemma Se $f\in L^1(\Omega)$ e per ogni $\varphi\in C_c^\infty(\Omega)$

(I) si ha
$$\int_\Omega f\cdot\varphi = 0$$
Allora $f\equiv 0$ q.o.

(II) se $\forall \varphi\in C_c^\infty(\Omega)$
$$\int_\Omega f\cdot\nabla\varphi = 0$$
Allora $\exists c\in\mathbb{R}: f\equiv c$ q.o.

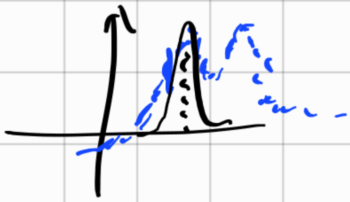
Per dimostrare questi lemmi ci servono dei prerequisiti tecnici.



Metodo 1 ACCENNATO

Uso i mollificatori:
$$(f*g)(x) = \underbrace{\int f(y)g(y-x)\,dy}$$

$$\int\rho_\varepsilon = 1$$



$f\in L^1$ $\qquad f_\varepsilon = f*\rho_\varepsilon$

$f_\varepsilon\in C^\infty$.

(I) $$\int f_\varepsilon\cdot\varphi = \int (f*\rho_\varepsilon)\cdot\varphi = \int f\cdot(\varphi*\rho_\varepsilon) = 0 \quad \forall\varphi\in C_c^\infty.$$

Dunque: $f_\varepsilon = 0$ $\qquad f\in L^1$ $\quad f_\varepsilon \xrightarrow{L^1} f$ quindi $f=0$

(II) $$\underline{\int f_\varepsilon\cdot\nabla\varphi} = \int (f*\rho_\varepsilon)\cdot\nabla\varphi = \int f\cdot(\nabla\varphi*\rho_\varepsilon) = \int f\cdot\nabla(\varphi*\rho_\varepsilon) = 0 \quad \forall\varphi$$

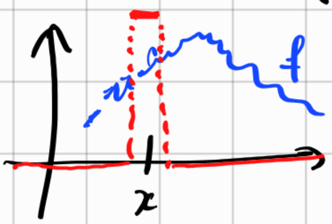
$\Downarrow$

$f_\varepsilon = c_\varepsilon\in\mathbb{R}$ $\qquad f_\varepsilon\xrightarrow{L^1} f$ $\qquad c_\varepsilon\to f$ $\qquad f = c\in\mathbb{R}$ $\qquad\square$.

Metodo 2: teorema di Lebesgue.

Teorema se $f \in L^1_{loc}(\Omega)$ per q.o. $x \in \Omega$ si ha:

$$f(x) = \lim_{\rho \to 0} \fint_{B_\rho(x)} f = \lim_{\rho \to 0} \int f \cdot \frac{\chi_{B_\rho(x)}}{|B_\rho(x)|}$$



(Ci serve)

Lemma (ricoprimento tipo Vitali) Sia X spazio metrico totalmente limitato (se $X \subseteq \mathbb{R}^n$ significa X limitato). Sia $\rho: X \to (0, +\infty)$. Allora esiste un insieme al più numerabile di punti x_k tali che:

1. le palle $B_{\rho(x_k)}(x_k)$ sono a 2 a 2 disgiunte
2. $X \subseteq \bigcup_k B_{3\rho(x_k)}(x)$.

def X totalmente limitato: $\forall \varepsilon > 0$ $\exists Y \subseteq X$ Y finito t.c.

$$\bigcup_{y \in Y} B_\varepsilon(y) \supseteq X.$$