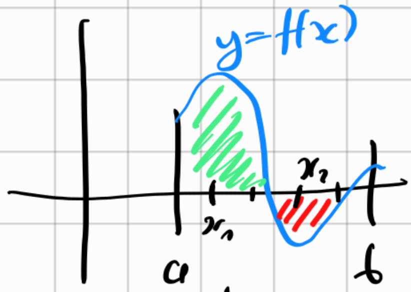
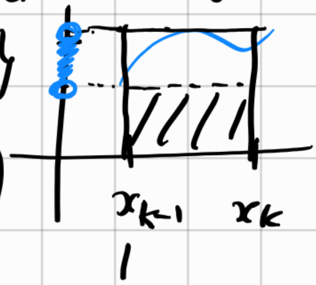
ANALISI MATEMATICA B
LEZIONE 56 - 24.2.2023

Integrale di Riemann



$\int_a^b f(x)dx$, f limitata

$P \subseteq [a,b]$ $\quad P = \{a = x_0 < x_1 < \dots < x_N = b\}$



$$S^*(f,P) = \sum_{k=1}^{N} \sup f([x_{k-1},x_k]) \cdot (x_k - x_{k-1})$$

$$S_*(f,P) = \sum_{k=1}^{N} \inf f([x_{k-1},x_k]) (x_k - x_{k-1})$$

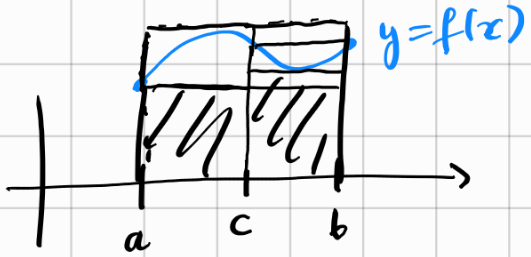
$$S^*(f) = \inf \{S^*(f,P) : P \text{ suddivisione di } [a,b]\}$$

$$S_*(f) = \sup \{S_*(f,P) : P \text{ suddivisione di } [a,b]\}$$

Se $S^*(f) = S_*(f)$ diremo che f è R-integrabile.

$$\int_a^b f$$

Criteri di integrabilità



$$S_*(f,\{a,b\}) \le S_*(f,\{a,c,b\}) \le S^*(f,\{a,c,b\}) \le S^*(f,\{a,b\})$$

Per induzione se $P \subseteq Q$

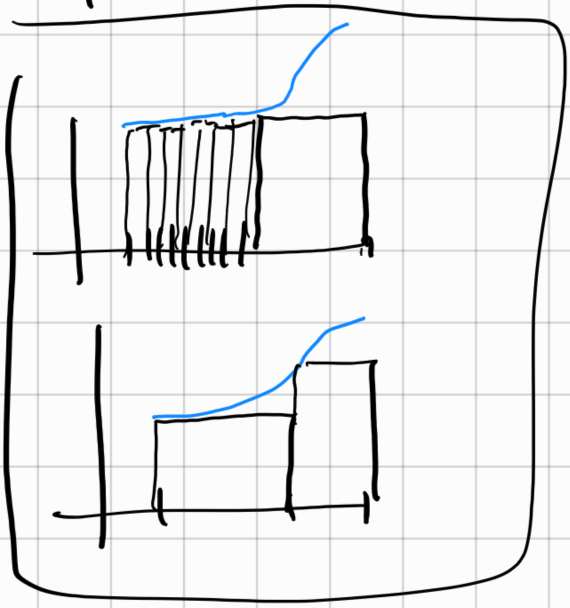
- $S_*(f,P) \le S_*(f,Q) \le S^*(f,Q) \le S^*(f,P)$

Lemma Se P e Q sono suddivisioni qualunque

$$S_*(f,P) \le S_*(f,P\cup Q) \le S^*(f,P\cup Q) \le S^*(f,Q)$$

$$S^*(f) = \inf_{P} S^*(f,P) \ge \sup_{P} S_*(f,P) = S_*(f)$$

$$S_*(f) \le S^*(f).$$



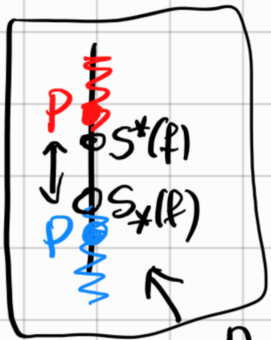
Teorema 1 f è R-integrabile su $[a,b]$

$f:[a,b]\to\mathbb{R}$, limitata.

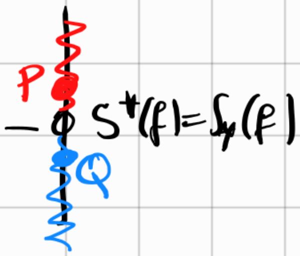
f è R-integrabile

$\Updownarrow$

$\forall \varepsilon>0 \quad \exists P$ suddivisione t.c. $\underline{\underline{S^*(f,P)-S_*(f,P) \le \varepsilon.}}$



non R-integr.



Dato $\varepsilon>0$

$$\exists P: \quad S^*(f,P) - S^*(f) \le \frac{\varepsilon}{2} \quad (1)$$

se f è integr.

$$\exists Q: \quad S_*(f) - S_*(f,Q) \le \frac{\varepsilon}{2} \quad (2)$$

$$S^*(f,P\cup Q) \overset{\text{lemma}}{\le} S^*(f,P) \overset{(1)}{\le} S^*(f)+\frac{\varepsilon}{2} \underset{\text{f è R-integr.}}{=} S_*(f)+\frac{\varepsilon}{2} \overset{(2)}{\le} S_*(f,Q)+\varepsilon \underset{\text{lemma}}{\le} S_*(f,P\cup Q)+\varepsilon$$

Viceversa:

$$\underbrace{S_*(f,P) \le \underbrace{S_*(f) \le S^*(f)}_{<\varepsilon} \le S^*(f,P)}_{<\varepsilon}$$

$\forall \varepsilon>0$

$$0 \le S^*(f)-S_*(f) \le \varepsilon$$

$\Downarrow$

$$S^*(f)=S_*(f).$$

Teorema 2 Se f è R-int. su $[a,b]$ allora $\exists$ P_n successione di suddivisioni t.c.

$$\lim_{n\to+\infty} S^*(f,P_n) = \int_a^b f.$$

$$\lim_{n\to+\infty} S_*(f,P_n) = \int_a^b f.$$

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dim Prendo $\varepsilon_n = \frac{1}{n}$. Per Teo 1 $\exists$ P_n

$$S^*(f,P_n) - S_*(f,P_n) < \frac{1}{n}$$

$$\underbrace{S_*(f,P_n) \le \underbrace{S_*(f) = \int_a^b f}_{<\frac{1}{n}} \underbrace{= S^*(f) \le S^*(f,P_n)}_{<\frac{1}{n}}}_{<\frac{1}{n}}$$

□

Sia $f:[a,b]\to\mathbb{R}$ limitata.

Viceversa Se esiste P_n succ. di suddivisioni

t.c. $S^*(f,P_n) - S_*(f,P_n) \longrightarrow 0$

Allora f è R-integrabile e

$$\int_a^b f = \lim S^*(f,P_n)$$

$$= \lim S_*(f,P_n)$$

dim

$\forall\varepsilon>0$ $\exists n$: $S^*(f,P_n) - S_*(f,P_n) < \varepsilon$.

per Teo 1 f è R-integrabile.

inoltre $S_*(f, P_n) \le \int f \le S^*(f, P_n)$

$S_*(f,P_n) \to \ell$, $S^*(f,P_n) \to \ell$

$\square$

$$\left[\begin{array}{ll} P_n = P(n) & P : \mathbb{N} \to \{\text{suddivisioni}\} \\ (a_n) & a : \mathbb{N} \to X \end{array}\right]$$

Esempio <u>Primo esempio di integrale.</u> $a = 0$ $b > 0$

$f : [0, b] \to \mathbb{R}$

f è integrabile ?

$f(x) = x^2.$

Quanto vale $\int_0^b x^2\, dx$?

(f è limitata per Weierstraß).

oppure $0 \le f(x) \le b^2$.

$$P_n = \left\{ x_0 = 0,\ x_1 = \frac{b}{n},\ x_2 = \frac{2b}{n}, \dots, \underline{x_k = k \cdot \frac{b}{n}}, \dots x_n = b \right\}$$

$[x_{k-1}, x_k]$

$$S^*(f, P_n) = \sum_{k=1}^{n} f(x_k) \cdot (x_k - x_{k-1})$$

$\uparrow$ f è crescente

$$= \sum_{k=1}^{n} \left(k \cdot \frac{b}{n}\right)^2 \cdot \frac{b}{n} = \left(\sum_{k=1}^{n} k^2\right) \frac{b^3}{n^3}$$

$$S_*(f, P_n) = \sum_{k=1}^{n} f(x_{k-1}) \cdot (x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \left((k-1)\frac{b}{n}\right)^2 \cdot \frac{b}{n} = \left(\sum_{k=1}^{n} (k-1)^2\right) \cdot \frac{b^3}{n^3}$$

$\uparrow$

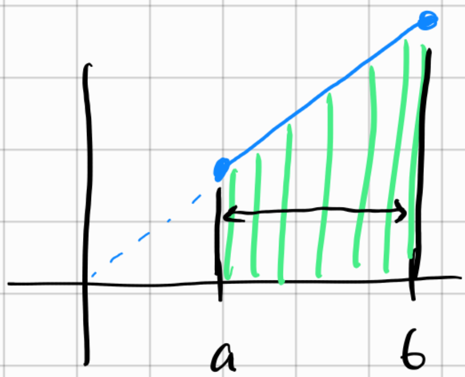
$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

$$S^*(f,P_n) = \frac{2n^3+3n^2+n}{6\cdot n^3}\, b^3 \longrightarrow \frac{b^3}{3}$$

$$S_*(f,P_n) = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6n^3}\, b^3 \longrightarrow \frac{b^3}{3}.$$

$\Rightarrow f$ è R-integrabile

e $\displaystyle\int_0^b f = \frac{b^3}{3}$ $\square$.



Esercizi

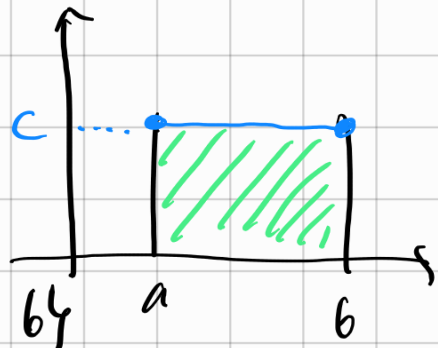
$$\int_a^b x\,dx \overset{?}{=} \frac{f(a)+f(b)}{2}\cdot(b-a)$$

area trapezio

Esercizio

$$\int_a^b c\,dx = c\cdot(b-a).$$

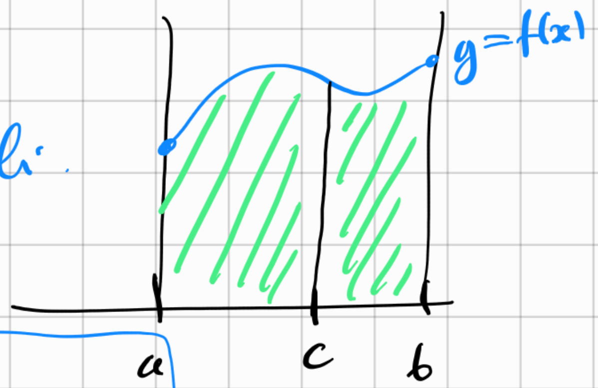
(Basta una unica suddivisione banale $P=\{a,b\}$)



Come è fatto $R([a,b]) = \{f:[a,b]\to\mathbb{R} : f$ limitata e R-integrabile$\}$

classe di funzioni.

Propietà delle fn. integrabili.



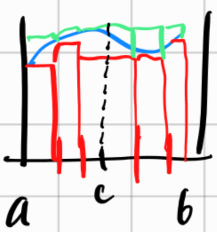
Teo (additività) $a\leq c\leq b$

$$\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx} \quad (*)$$

$$f\in R([a,b]) \iff f\in R([a,c]) \cap R([c,b])$$

e vale (*)

dim "$\Rightarrow$"



Ho $P \rightsquigarrow P' = P\cup\{c\}$

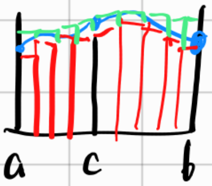
$\downarrow$

$P_1 = P'\cap[a,c]$

$P_2 = P'\cap[c,b]$

$$\underbrace{S^*_*(f,P')}_{\int_a^b f} = \underbrace{S^*_*(f,P_1)}_{\int_a^c f} + \underbrace{S^*_*(f,P_2)}_{\int_c^b f}$$

"$\Leftarrow$"



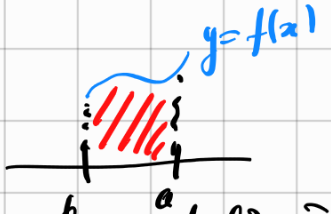
$\forall\varepsilon>0\ \exists P_1\subseteq[a,c]\ \exists P_2\subseteq[c,b]$

$\Downarrow$

$P = P_1\cup P_2.$ $\square$

Oss $\quad a\le c\le b \qquad \int_a^b f = \int_a^c f + \int_c^b f$

$$\int_a^c f = \int_a^b f - \int_c^b f$$



Notazione: Se $a>b$ poniamo per definizione $f:[b,a]\to\mathbb{R}$

$$\boxed{\int_a^b f} = -\boxed{\int_b^a f}$$

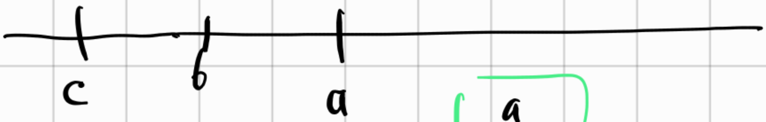
per definizione — lo abbiamo definito

Con questa notazione $\int_a^b f = \int_a^c f + \int_c^b f$

in qualunque ordine siano a,b,c.

Es

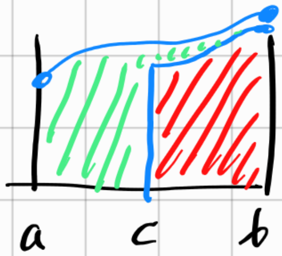
$c < b < a$



$$\int_c^a f = \int_c^b f + \boxed{\int_b^a f} \quad \leftarrow \text{Teorema}$$

Notazione → $$\int_a^b f = -\int_b^a f = \int_c^b f - \int_c^a f \underset{\text{Notazione}}{=} \int_c^b f - \left(-\int_a^c f\right)$$

$$= \int_a^c f + \int_c^b f \qquad \square$$



$$\int_a^c f = \int_a^b f - \int_c^b f$$

$$= \int_a^b f + \int_b^c f \qquad \square$$