27/05/2009

$$\boxed{x y' = 3y - 2x^2 - 1}$$

$$x y' = 2x + \frac{x^2 + x + 1}{x^2 + 1}$$

$$y' = \frac{y}{1 - y^2} x$$

$$\begin{cases} x y' = 3y - 2x^2 - 1 \\ y(1) = y_0 \end{cases}$$

$x > 0$

$$y' = \frac{3y}{x} - \frac{2x^2 + 1}{x}$$

$a(x) = \frac{3}{x}$ $\quad b(x) = -\frac{2x^2 + 1}{x}$ $\quad$ su $\{x > 0\}$

$$A(x) = 3 \int_1^x \frac{1}{t} \, dt = \ln(t^3) \Big|_1^x = \ln(x^3)$$

$$y(x) = x^3 \left(y_0 - \int_1^x \frac{1}{t^3} \frac{2t^2 + 1}{t} \, dt \right) =$$

$$x^3\left(y_0 - \int_1^x \left(\frac{2}{t^2} + \frac{1}{t^4}\right)dt\right) = x^3\left(y_0 - \left[-\frac{2}{t} - \frac{1}{3t^3}\right]_1^x\right) =$$

$$x^3\left(c + \frac{2}{x} + \frac{1}{3x^3}\right) \qquad \text{dove } c = y_0 - \frac{7}{3}$$

$$= cx^3 + 2x^2 + \frac{1}{3} = \frac{1}{3} + x^2(cx+2)$$

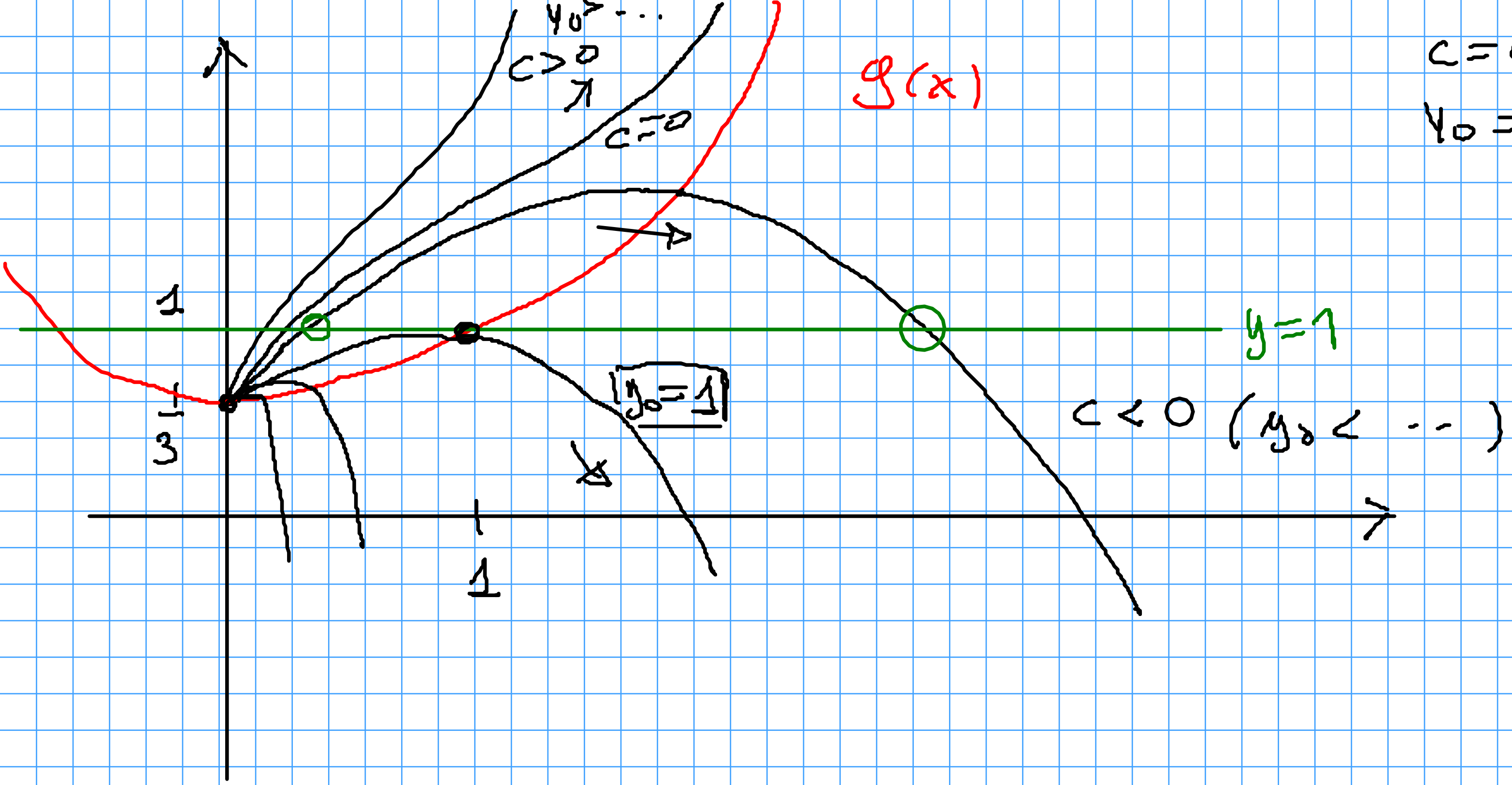
LIMITI

- $\lim_{x\to 0^+} y(x) = \frac{1}{3}^+$ lo vedo dopo

$$\lim_{x\to+\infty} y(x) = \begin{cases} +\infty & c \geq 0 \\ -\infty & c < 0 \end{cases}$$

MONOTONIA STUDIO il segno di y' = segno di $3y - (2x^2+1)$

$$3y - (2x^2+1) > 0 \iff y > \frac{2x^2+1}{3} =: g(x)$$



$c=0 \iff$ $y_0 = \frac{7}{3}$

$y_0 =$

STUDIO DELLE SOL. DI $y(x) = 1$

quante volte la sol. $y(x)$ attraversa la retta $y=1$

CONVIENE TROVARE l'intersezione tra il grafico di $y=g(x)$ e $y=1$.

$$\frac{2x^2+1}{3} = 1 \iff 2x^2+1 = 3 \iff x^2 = 1 \qquad x = \pm 1$$

- la soluzione con $y_0 = 1$ è "tangente" a $y=1$ $\to$ 1 sol.

- le soluzioni con $y_0 < 1$ sono sotto la retta $y=1$ $\Rightarrow$ 0 sol.
- se $1 \leq y_0 < \frac{7}{3}$ ci sono due sol.
- se $y_0 \geq \frac{7}{3}$ c'è una sol.

$\Longrightarrow$ La risposta giusta all'ultima domanda è $\boxed{1 < y_0 < 7/3}$

ESERCIZIO

Studiamo le sol. di

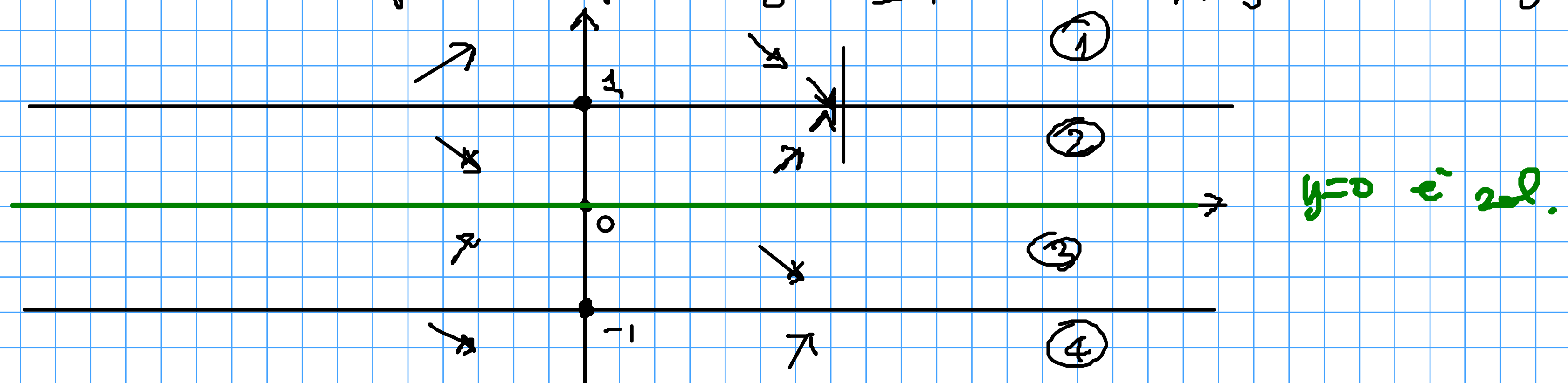
$$y' = \frac{y}{1-y^2}\, x = F(x,y)$$

(EQ. VARIABILI SEPARABILI)

$y' = A(y)\, B(x)$

$A(y) = \frac{y}{1-y^2}$ $\qquad$ $B(x) = x$

A definita per $y \neq \pm 1$ $\qquad$ $A(y)=0$ se $y=0$



Applichiamo le formule: fisso x_0, $y_0 \neq \pm 1$

risolvo l'eq. con la condizione $y(x_0) = y_0$

$$y' \frac{1-y^2}{y} = x \qquad \text{INTEGRO TRA } x_0 \text{ e } x$$

$$\int_{x_0}^{x} \frac{1-y(t)^2}{y(t)} y'(t)\, dt = \int_{x_0}^{x} t\, dt = \frac{x^2}{2} - \frac{x_0^2}{2}$$

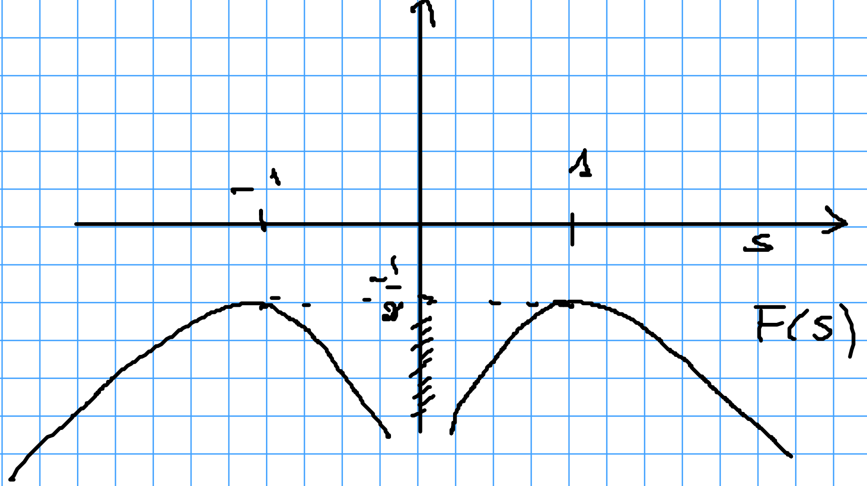
$$\int_{y_0}^{y(x)} \frac{1-s^2}{s} ds = \int_{y_0}^{y(x)} \left(\frac{1}{s} - s\right) ds = \left[\ln|s| - \frac{s^2}{2}\right]_{y_0}^{y(x)}$$

Posto $F(s) = \ln|s| - \frac{s^2}{2}$ Trovo la relazione

$$F(y(x)) = F(y_0) + \frac{x^2}{2} - \frac{x_0^2}{2}$$

VADO A DISTINGUERE I VARI INTERVALLI IN CUI

F è invertibile. Prima di tutto facciamo il grafico di F



$$\lim_{s\to0^+} F(s) = -\infty$$

$$\lim_{s\to+\infty} F(x) = -\infty$$

$$F'(s)=0 \iff s=\pm 1$$

$$F(\pm 1) = -\frac{1}{2}$$

<u>CASO 1</u> $\quad y_0>0 \quad$ Devo invertire F su $[1,+\infty[$

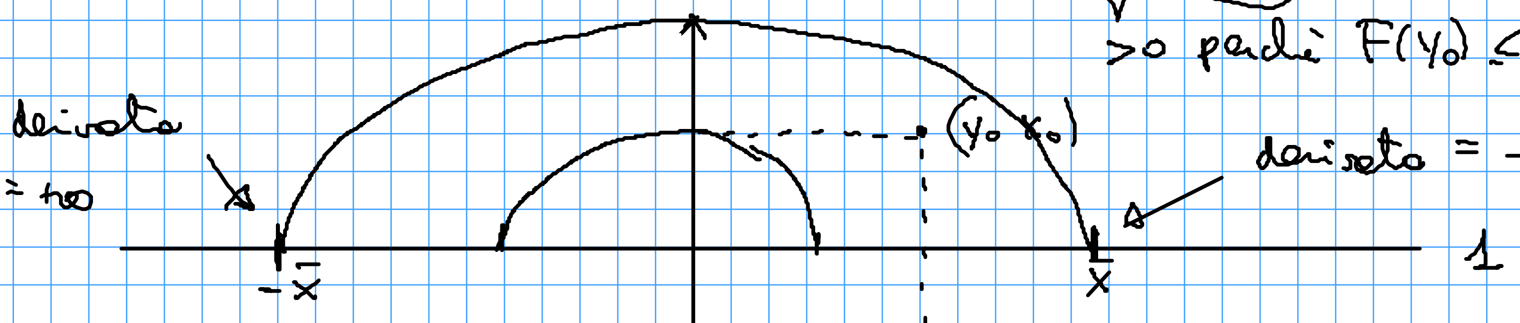
$\Rightarrow \quad F^{-1}: \;]-\infty, -\frac{1}{2}] \to [1, +\infty[$ decrescente

e la relazione $\quad y(x) = F^{-1}\left(F(y_0) - \frac{x_0^2}{2} + \frac{x^2}{2}\right)$

ha senso fino a quando $\quad F(y_0) - \frac{x_0^2}{2} + \frac{x^2}{2} \in \;]-\infty, -\frac{1}{2}]$

cioè quando $\quad \frac{x^2}{2} \le \underbrace{-\frac{1}{2} + \frac{x_0^2}{2} - F(y_0)}_{>0 \text{ perché } F(y_0) \le -\frac{1}{2}} \iff -\bar{x} \le x \le \bar{x}$

$$\left(\bar{x} = \sqrt{-1 + x_0^2 - 2F(y_0)}\right)$$



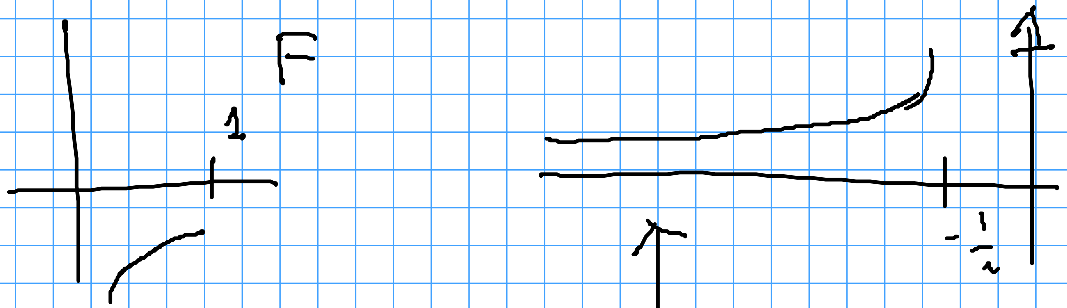
CASO 2 $\quad 0 < y_0 < 1$ $\qquad$ INVERTO $\quad F :]0, 1] \to]-\infty, -\frac{1}{2}]$

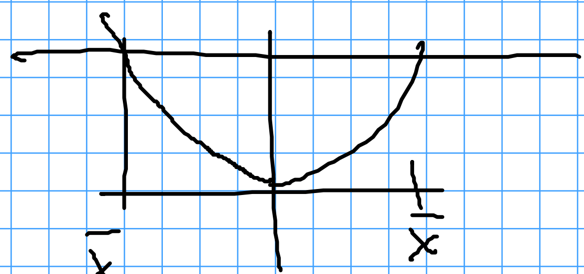
$F^{-1} :]-\infty, -\frac{1}{2}] \to]0, 1]$ CRESCENTE

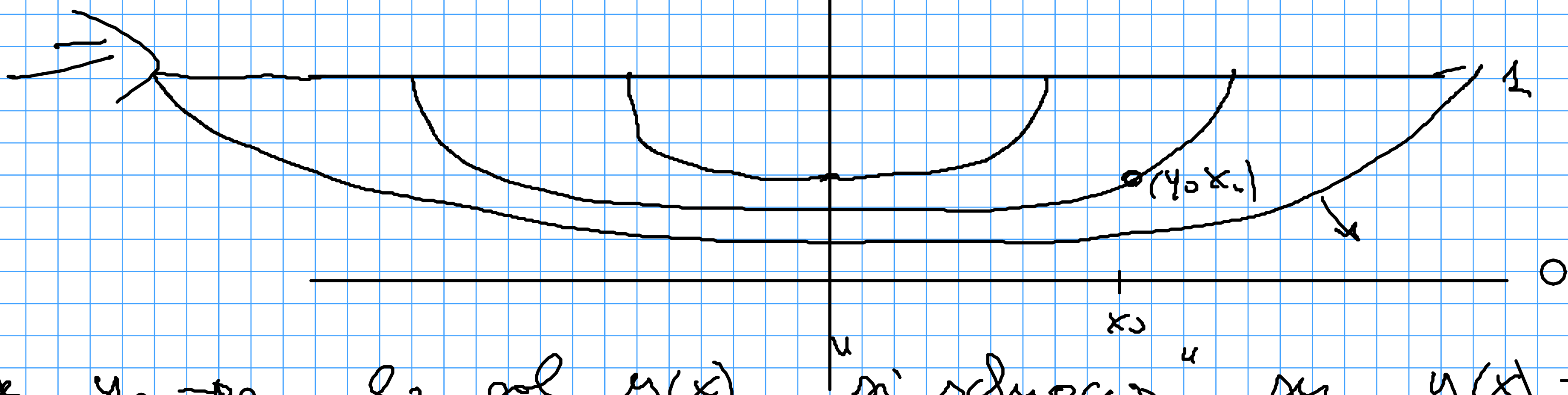
e la sol $\quad y(x) = F^{-1}\left(F(y_0) - \frac{x_0^2}{2} + \frac{x^2}{2} \right)$

esiste fino a quando $\quad F(y_0) - \frac{x_0^2}{2} + \frac{x^2}{2} \leq -\frac{1}{2}$

STAVOLTA







(per $y_0 \to 0$ la sol. $y(x)$ "si schiaccia" su $y(x) = 0$

PROVARE L'EQ (SIMILE)

$$y' = \frac{yx}{y^2-1}$$

Altro esercizio sulle lineari:

$$x y' = 2y + \frac{x^2+x+1}{x^2+1} \qquad \underline{x > 0} \qquad y(1) = y_0$$

Metto in forma normale

$$y' = \frac{2}{x} y + \frac{x^2+x+1}{x(x^2+1)}$$, APPLICO LA FORMULA

$$y(x) = x^2 \left(y_0 + \int_1^x \frac{t^2+t+1}{t^3(t^2+1)}\, dt \right) = \circledast$$

Riduco l'integrando "in fratti semplici"

$$\frac{t^2+t+1}{t^3(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{Dt+E}{t^2+1} =$$

$$\frac{A(t^4+t^2) + B(t^3+t) + C(t^2+1) + Dt^4 + Et^3}{t^3(t^2+1)}$$

$$\begin{cases} A + D = 0 \\ B + E = 0 \\ A + C = 1 \\ B = 1 \\ C = 1 \end{cases} \qquad \begin{cases} A = 0 \\ B = 1 \\ C = 1 \\ D = 0 \\ E = -1 \end{cases} \longrightarrow \frac{1}{t^2} + \frac{1}{t^3} - \frac{1}{t^2+1}$$

$$\circledast = x^2\left(y_0 + \int_1^x \left(\frac{1}{t^2} + \frac{1}{t^3} - \frac{1}{t^2+1}\right) dt\right) =$$

$$x^2\left(y_0 + \left[-\frac{1}{t} - \frac{1}{2t^2} - \operatorname{arctg}(t)\right]_1^x\right) =$$

$$x^2\left(y_0 + \frac{3}{2} + \frac{\pi}{4} - \frac{1}{x} - \frac{1}{2x^2} - \operatorname{arctg}(x)\right) =$$

$$x^2\left(c - \frac{1}{x} - \frac{1}{2x^2} - \operatorname{arctg}(x)\right) \qquad \text{dove } \boxed{c = y_0 + \frac{3}{2} + \frac{\pi}{4}}$$

LIMITI

$$\lim_{x \to 0^+} y(x) = \lim_{x\to 0^+} cx^2 - x - \frac{1}{2} - x^2 \operatorname{arctg}(x) = -\frac{1}{2}^-$$

$$\lim_{x\to +\infty} y(x) = \begin{cases} +\infty & c > \frac{\pi}{2} \iff y_0 > \frac{\pi}{4} - \frac{3}{2} \\ ? & c = \frac{\pi}{2} \iff y_0 = \text{"} \quad \text{"} \\ -\infty & c < \frac{\pi}{2} \iff y_0 < \text{"} \quad \text{"} \end{cases}$$

$$c = \frac{\pi}{2} \iff y_0 + \frac{3}{2} + \frac{\pi}{4} = \frac{\pi}{2} \iff y_0 = \frac{\pi}{4} - \frac{3}{2}$$

CASO $c = \frac{\pi}{2}$

$$\lim_{x\to+\infty} x^2\left(\frac{\pi}{2} - \operatorname{arctg}(x) - \frac{1}{x} - \frac{1}{2x^2}\right) = \left(t = \frac{1}{x}\right)$$

$$\lim_{t\to 0^+} \frac{\left(\frac{\pi}{2} - \operatorname{arctg}\left(\frac{1}{t}\right) - t - \frac{t^2}{2}\right)}{t^2} \overset{\text{Hôpital}}{=} \lim_{t\to 0^+} \frac{-\frac{1}{1+\frac{1}{t^2}}\left(-\frac{1}{t^2}\right) - 1 - t}{2t}$$

$$= \lim_{t \to 0} \frac{\frac{1}{t^2+1} - 1 - t}{2t} = \lim_{t \to 0^+} \frac{\cancel{1} - t^2 - \cancel{1} - t(t^2+1)}{(t^2+1)\,2t} =$$

$$\lim_{t \to 0^+} \frac{-t - t^2 - 1}{(t^2+1) \cdot 2} = -\frac{1}{2}$$

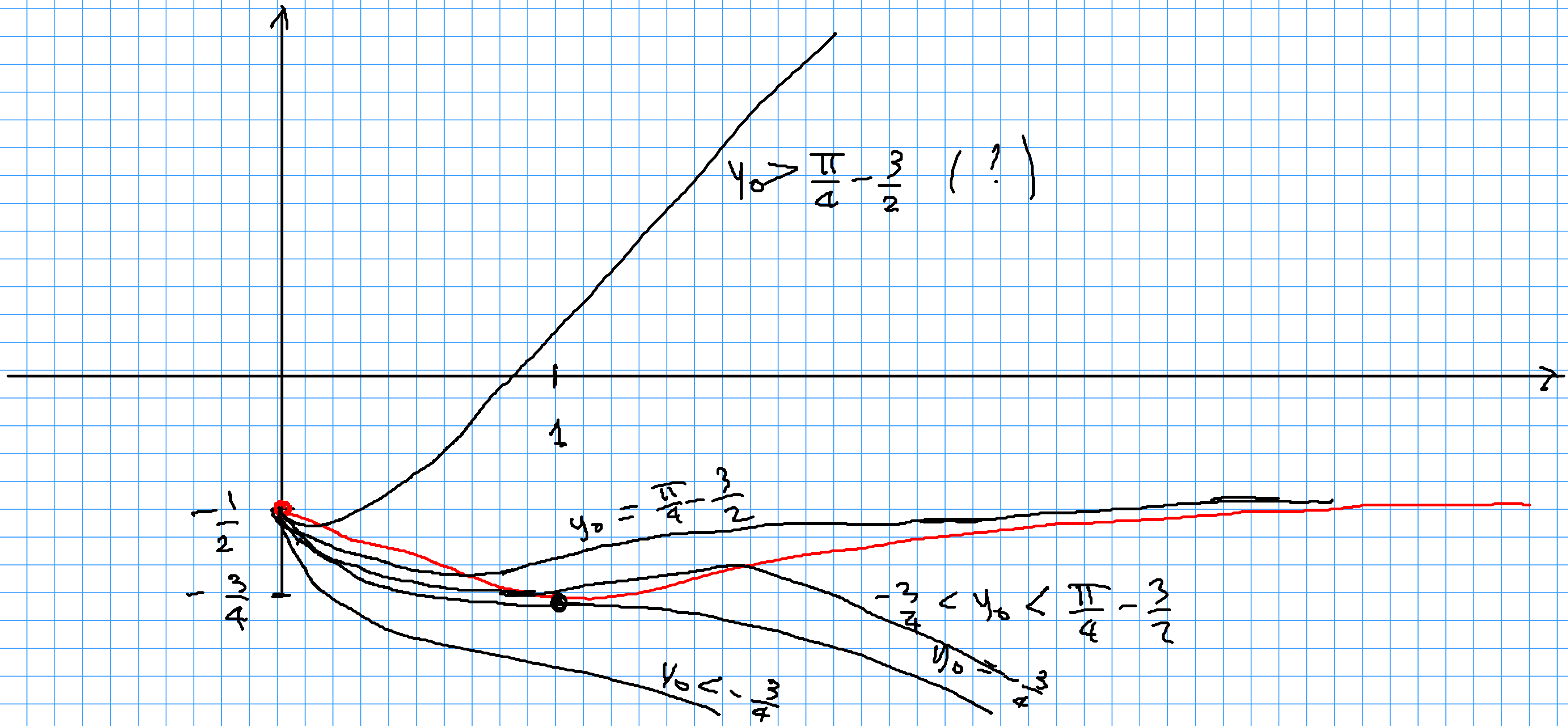
MONOTONIA $\quad y' > 0 \iff 2y > -\frac{x^2+x+1}{x^2+1}$

$$\iff y > -\frac{1}{2} \frac{x^2+x+1}{x^2+1} = g(x) \qquad \text{STUDIO } g(x) \ (\text{sulle } x>0)$$

$$\lim_{x \to \pm\infty} g(x) = -\frac{1}{2} \qquad g(0) = -\frac{1}{2} \qquad \begin{array}{l} \cancel{2x^3} + x^2 + 2x + 1 \\ - \ \cancel{2x^3} - 2x^2 - 2x \end{array}$$

$$g'(x) = -\frac{1}{2} \frac{(2x+1)(x^2+1) - (x^2+x+1)2x}{(x^2+1)^2} = -\frac{1}{2} \frac{-x^2+1}{(x^2+1)^2}$$

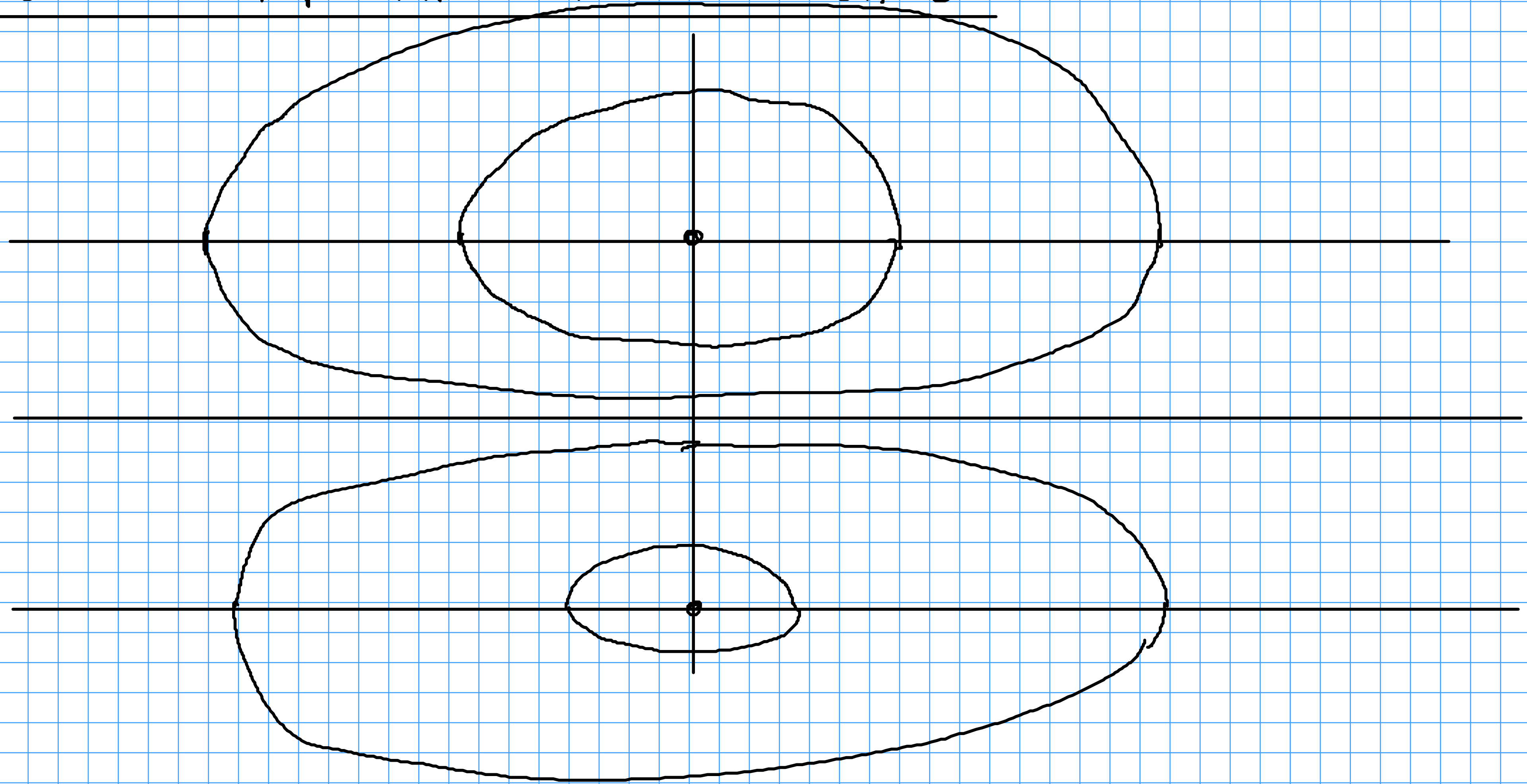
$$= \frac{x^2-1}{2(x^2+1)^2} \qquad g(1) = -\frac{3}{4}$$



$y_0 > \frac{\pi}{4} - \frac{3}{2}$ (!)
1
$-\frac{1}{2}$
$-\frac{3}{4}$
$y_0 = \frac{\pi}{4} - \frac{3}{2}$
$-\frac{3}{4} < y_0 < \frac{\pi}{4} - \frac{3}{2}$
$y_0 = -\frac{3}{4}$
$y_0 < -\frac{3}{4}$

LASCIAMO I CASI 3 e 4 — VIENE

UN COMPORTAMENTO SPECULARE



DANDO UNA DEFINIZIONE OPPORTUNA (NON PIÙ $y = y(x)$

MA DI TIPO "PARAMETRICO" $y = y(t)$ $x = x(t)$)

$\Rightarrow$ SI PUÒ RITENERE CHE LE FIGURE SOPRA

SIANO SOLUZIONI

$$y' = \frac{x}{y-1}$$

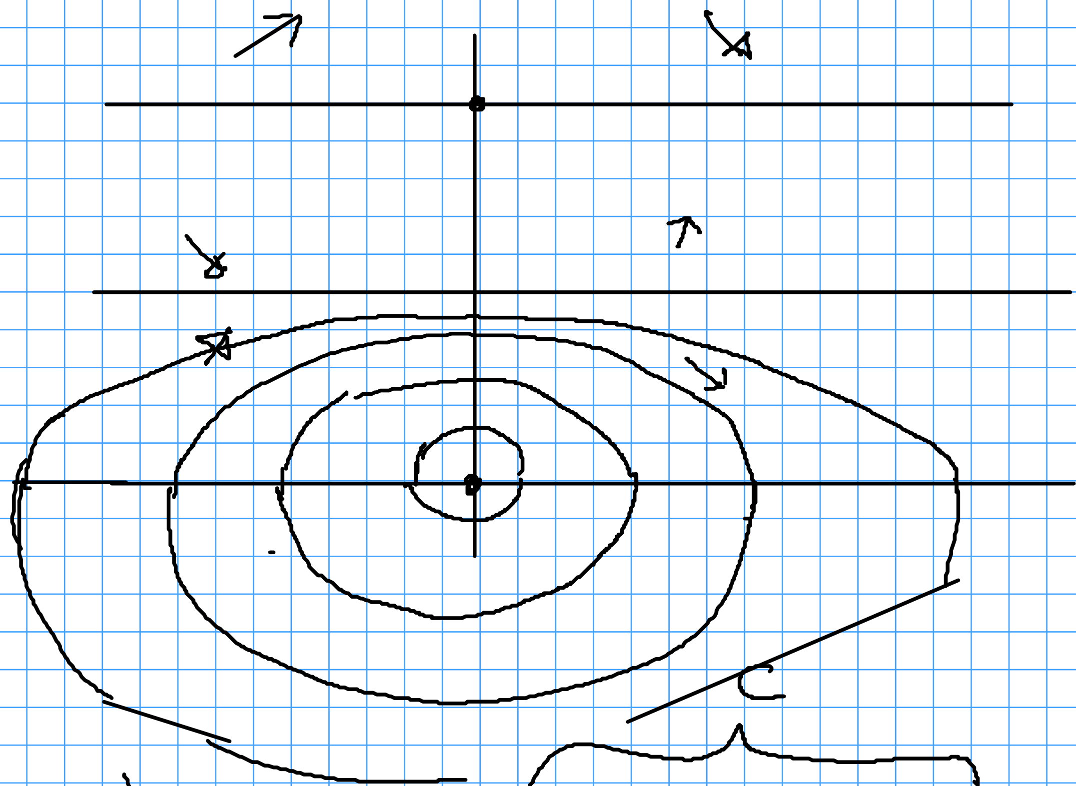
$$\boxed{y' = \frac{y}{1-y^2}\, x}$$

$$\int_{x_0}^{x} y'(x)\, \frac{(1-y^2)}{y}\, dx = \int_{x_0}^{x} t\, dt$$

$$\int_{y_0}^{y(x)} \frac{1-s^2}{s}\, ds = \frac{x^2}{2} - \frac{x_0^2}{2}$$

$$\parallel$$

$$\left[\ln|s| - \frac{s^2}{2}\right]_{y_0}^{y(x)} = \ln(|y(x)|) - \frac{y(x)^2}{2} = \overbrace{\underbrace{\ln|y_0| - \frac{y_0^2}{2}}_{F(y_0)} - \frac{x_0^2}{2}}^{C} + \frac{x^2}{2}$$

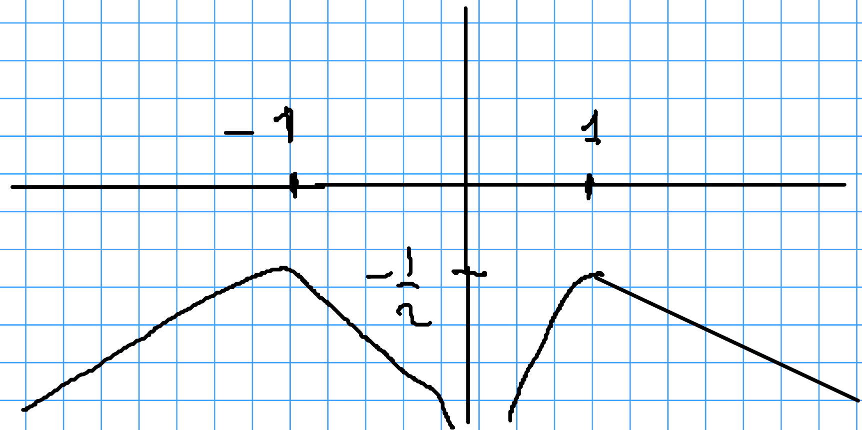


$$F(s) := \ln|s| - \frac{s^2}{2}$$

$\underline{s>0}$ $\qquad F(0) = -\infty \qquad F'(s) = \frac{1}{s} - s = \frac{1-s^2}{s}$

$F'(s) = 0 \iff s = 1$

$F(1) = -\frac{1}{2}$



PARI

F^{-1}:

$]-\infty, -1] \to]-\infty, -\frac{1}{2}] = J_1$ CRESCENTE ①

$[-1, 0] \to]-\infty, -\frac{1}{2}[= J_2$ DECRESCENTE

$]0, 1] \to]-\infty, -\frac{1}{2}[= J_3$ CRESCENTE

$[1, +\infty[\to]-\infty, \frac{1}{2}] = J_4$ DECRESCENTE

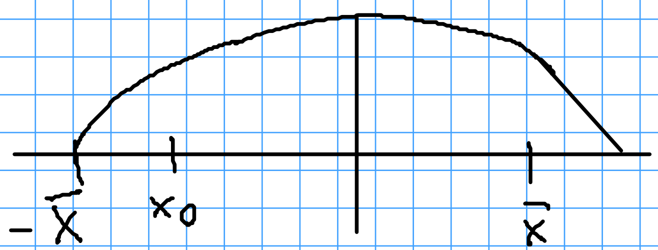
$\boxed{y_0 < -1}$

CASO 1

$$y(x) = F^{-1}\left(C + \frac{x^2}{2}\right)$$

$$F(y_0) - \frac{x_0^2}{2} + \frac{x^2}{2} \le -\frac{1}{2} \iff x^2 \le + x_0^2 - (2F(y_0)+1)$$

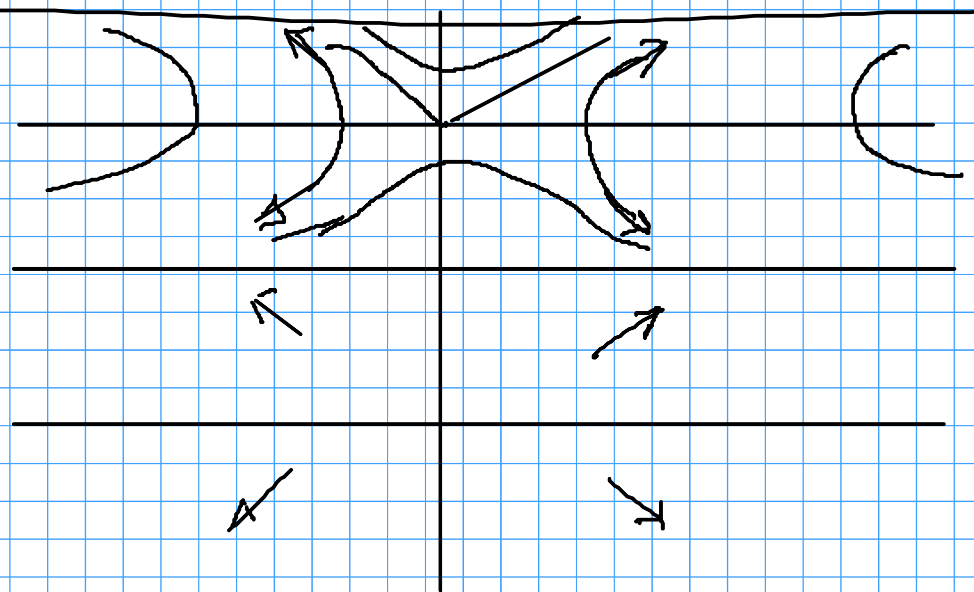
$$\left(x = x_0 \quad \frac{x_0^2}{2} \le -\frac{1}{2} + \frac{x_0^2}{2} - F(y_0) \iff F(y_0) \le -\frac{1}{2} \quad \text{ok}\right)$$



$-1 \le y_0 < 0$

$$\frac{x^2}{2} - \frac{x_0^2}{2} + F(y_0) < \frac{1}{2} \qquad x^2 \le 1 + x_0^2 - 2F(y)$$

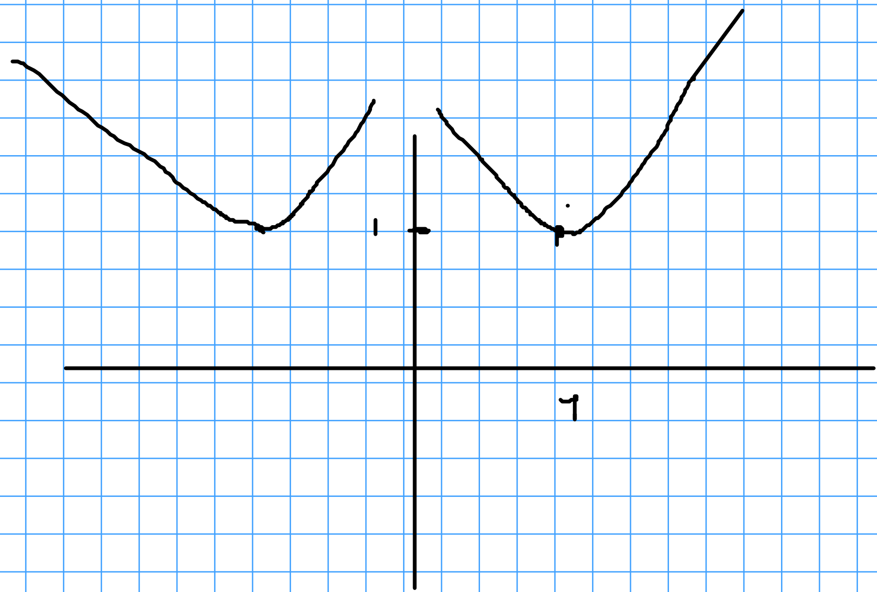
$$y' = \frac{xy}{y^2-1}$$



$$y'(y^2-1)/y = x$$

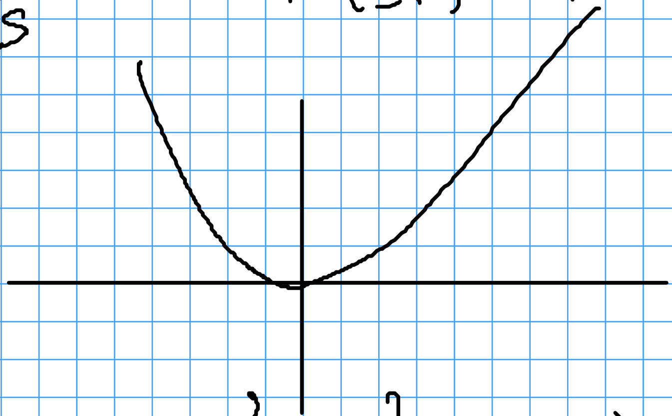
$$2\int_{y_0}^{y(x)} \frac{s^2-1}{s}\, ds = x^2 - x_0^2$$

$$\left[s^2 - \ln(s^2)\right]_{y_0}^{y(x)} = x^2 - x_0^2$$



$$F(s) = s^2 - \ln(s^2)$$

$$F' = \frac{s^2-1}{s} \qquad F(\pm 1) = 1$$



$y_0 > 1$ $\qquad x^2 - x_0^2 + F(y_0) \geq 1 \iff x^2 \geq x_0^2 + 1 - F(y_0)$

$$x_0^2 + 1 - F(y_0) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$