Report on the research activity carried out within the Research Grant "Matrix Analysis, Algorithms and Applications" at Dipartimento di Matematica, Università di Pisa

The main goal of this research project concerned the analysis of Quasi-Toeplitz (QT) matrices and their application to the design and analysis of algorithms for different computational problems.

A QT matrix A is a semi-infinite matrix that can be written as a sum of a Toeplitz matrix $T(a)$ and a compact operator K, i.e., $A = T(a) + K$, where $a = (a_i)_{i\in\mathbb{Z}}$, $\sum_{i\in\mathbb{Z}} |a_i| < \infty$, and $T(a) = (t_{i,j})_{i,j\in\mathbb{Z}^+}$, $t_{i,j} = a_{j-i}$. Matrices of this kind are encountered in several applications, in particular in the mathematical modeling of random walks in the quarter plane, and represent linear operators acting in the space ℓ^2.

The project originated by a previous collaboration with the group of Numerical Analysis of the Department of Mathematics, University of Pisa, where QT matrices have been applied in order to design effective algorithms for the numerical solution of *Random Walks in the Quarter Plane.* In fact, in the paper [4], the technology of QT matrices was applied to compute the minimal non-negative solution of matrix equations of the kind $AX^2 + BX + C = X$, where A, B, C are nonnegative QT matrices such that $A + B + C$ is stochastic. More specifically, fixed-point iterations were introduced and their convergence speed was analyzed.

Starting from this achievement, where the efforts were put in the analysis of the problem in the set ℓ^∞, due to the specificity of the physical model, we planned to study the solution of different matrix equations in different contexts. In particular, we considered the problem of defining and computing the geometric mean of a set $A_1, \dots, A_m$ of QT matrices that are symmetric and positive definite. This issue is encountered in some problems of radar detections in engineering where the matrices represent approximate measures of physical objects that must be averaged by respecting some conditions typical of the geometric mean. In these applications, matrices are finite and have the Toeplitz structure. Our belief was that computing the mean of infinite QT matrices is the right way to find good approximations in the finite case, from the point of view both of the computational complexity and of the accuracy of the approximation.

In the first 6 months, we have pursued this idea working at the extensions of the main available definitions of matrix geometric means to the case of QT operators in ℓ^2. In fact, if $a(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$ is real valued and essentially bounded, then $T(a)$, as well as, $A = T(a) + K$ represent bounded self-adjoint operators on ℓ^2. We consider the case where a is a continuous function, where quasi-Toeplitz matrices coincide with a classical Toeplitz algebra, and the case where a is in the Wiener algebra, that is, has absolutely convergent Fourier series.

We have proved that if $a_1, \dots, a_p$ are continuous and positive functions, or are in the Wiener algebra with some further conditions, then the means of geometric type, such as the Ando-Li-Mathias, the Nakamura-Bini-Meini-Poloni, and the Karcher mean of quasi-Toeplitz positive definite matrices associated with $a_1, \ldots, a_p$, are quasi-Toeplitz matrices associated with the geometric mean $(a_1 \cdots a_p)^{1/p}$, which differ only by the compact correction. We have shown by a wide set of numerical tests that these operator means can be practically approximated by extending the known algorithms to the QT case and proving their convergence. The effectiveness of this implementation also relies on the efficiency of the Matlab Toolbox `CQT-Toolbox` recently made available in the literature by the Pisa research group.

By using functional calculus, we have extended the main matrix functions $f(x)$ of interest, such as square root, p-root, logarithm and exponential, to the case of QT matrices, provided that the spectrum of the matrix is in the domain of definition of $f(x)$.

The results of this part of the research have been obtained in collaboration with Bruno Iannazzo from the University of Perugia, and have been collected in [2].

Then the attention has been addressed to the Karcher Mean and to the Power Mean of positive definite QT matrices. The Karcher Mean is defined as the solution of a nonpolynomial matrix equation involving the matrix logarithm function, while the Power Mean is defined as the solution of a matrix equation involving the geometric mean of suitable pairs of matrices. After showing that the Power Mean of quasi-Toeplitz matrices is a quasi-Toeplitz matrix, we have obtained a first algorithm based on the fact that the Karcher Mean is the limit of a family of power means. A second algorithm, that is shown to be more effective, is based on a generalization to the infinite-dimensional case of a reliable algorithm for computing the Karcher mean in the finite-dimensional case. Numerical tests show that the Karcher mean of infinite-dimensional quasi-Toeplitz matrices can be effectively approximated through a finite number of parameters.

The result of this research collaboration concerning the power mean and the Karcher mean, has led to the paper [3] and has been presented by Jie Meng at the international conference "*5th conference on Geometric Science of Information, Sorbonne University*, Paris, 21 July 2021 - 23 July 2021".

In the second part of the research activity, we have considered the problem of computing the eigenvalues of a QT matrix $A = T(a) + K$, that is, the complex numbers λ for which there exists a $v \in \ell^2 \setminus \{0\}$ such that $Av = \lambda v$. The problem can be approached in terms of the connected components formed by the set $\mathbb{C} \setminus a(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$ and $a(\mathbb{T})$ denotes the image of $\mathbb{T}$ through the action of the function $a(z)$. According to the value of the winding number of $a(z)$ at the points in each component, we may find out if a given component contains a continuous set of eigenvalues or if it contains isolated eigenvalues.

In the case where $a(z)$ is a Laurent polynomial and the compact correction K has a finite number of nonzero entries, we have reformulated the problem in terms of a *finite nonlinear* generalized eigenvalue problem. Observe that any QT matrix can be approximated to any accuracy in this set of finitely representable matrices.

Two versions of this formulation are given. The "Vandermonde" formulation expresses the matrices involved in the finite problem in terms of a Vandermonde matrix formed with the zeros of modulus less than 1 of the Laurent polynomial $a(z) - \lambda$, where λ is the sought eigenvalue. The "Frobenius" version expresses the problem in terms of a Frobenius matrix associated with the factor of the polynomial $a(z) - \lambda$ having roots of modulus less than 1.

For each of the two versions, we have provided different methods based on fixed point iteration, on Newton's iteration applied to a determinantal version and applied to a version given in terms of eigenvalues.

We have proved that the eigenvalues of the finite truncation of the QT matrix A at size N are good starting approximations to the isolated eigenvalue of A. This has motivated a heuristics that practically allows to effectively compute all the eigenvalues of A.

The algorithms have been implemented and tested on a wide variety of cases. The results of the numerical experiments performed so far indicate the reliability and the efficiency of the algorithms. We plan to insert these algorithms in the previously mentioned package `CQT-Toolbox` that is a Matlab Toolbox for managing with QT matrices.

This part of research has been carried out with Bruno Iannazzo, Beatrice Meini, and Leonardo Robol.

References:

[1] D.A. Bini, B. Iannazzo, B. Meini, J. Meng, L. Robol, "Computing eigenvalues of semi-infinite QT matrices: an integration to the CQT-Toolbox", in preparation.

[2] D.A. Bini, B. Iannazzo, J. Meng, "Geometric means of quasi-Toeplitz matrices", arXiv:2102.04302, 2021, submitted for publication.

[3] D.A. Bini, B. Iannazzo, J. Meng, "Algorithms for approximating means of semi-infinite quasi-Toeplitz matrices", Proceed. of 5th conference on Geometric Science of Information, Sorbonne University, Paris, July 2021.

[4] D.A. Bini, B. Meini, J. Meng," Solving quadratic matrix equations arising in random walks in the quarter plane", SIAM J. Matrix Anal. Appl. 2021.

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