

# An Introduction to Mean Curvature Flow

Carlo Mantegazza

ABSTRACT. In this series of lectures I will introduce the mean curvature flow of a compact hypersurface in the Euclidean space with particular attention to the cases of curves and surfaces. The basic properties and the main analytic and geometric techniques used in the analysis of this flow will be discussed, for instance maximum and comparison principles. Moreover, I will present the fundamental Huisken's monotonicity formula and the Harnack inequality by Hamilton, which are the key tools in the study of the singularity formation.

## Introduction

Let  $\varphi_0 : M \rightarrow \mathbb{R}^{n+1}$  be a smooth immersion of an  $n$ -dimensional smooth manifold in the Euclidean space. The evolution of  $M_0 = \varphi_0(M)$  by mean curvature is a smooth one-parameter family of immersions  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$  satisfying

$$\begin{cases} \frac{\partial}{\partial t} \varphi(p, t) = H(p, t) \nu(p, t) \\ \varphi(p, 0) = \varphi_0(p) \end{cases}$$

where  $H(p, t)$  and  $\nu(p, t)$  are respectively the mean curvature and the unit normal of the hypersurface  $M_t = \varphi_t(M)$  at the point  $p \in M$ , where  $\varphi_t = \varphi(\cdot, t)$ .

It can be checked that  $H(p, t) \nu(p, t) = \Delta_{g(t)} \varphi(p, t)$ , where  $\Delta_{g(t)}$  is the Laplace–Beltrami operator on  $M$  associated to the metric  $g(t)$ , induced by the immersion  $\varphi_t$ . Then, the mean curvature flow may be regarded as a sort of geometric heat equation, in particular it can be shown that it is a parabolic problem and has a unique solution for small time. In addition, the solutions satisfy comparison principles and derivatives estimates similar to the case of parabolic partial differential equations in the Euclidean space.

On the other hand, the mean curvature flow is not really equivalent to a heat equation, since the Laplace–Beltrami operator evolves with the hypersurface itself. In particular, in contrast to the classical heat equation, this flow is described by a nonlinear (quasilinear) evolution system of partial differential equations and the solutions exist in general only in a finite time interval.

Mean curvature flow occurs in the description of the evolution of the interfaces in several multiphase physical models (see e.g. [19, 21]). One can indeed date the “genesis” of the subject to the paper of Mullins [19]. The main reason for this is the property that it is the gradient-like flow of the *Area* functional and therefore it arises naturally in problems where a surface energy is relevant. From a physical point of view, it would be interesting also to consider the “hyperbolic” motion by mean curvature, that is, the evolution problem  $\partial_t^2 \varphi = H\nu$ , but very few results are present in the literature at the moment. Algorithms

based on the mean curvature flow has been also developed extensively in the field of automatic treatment of digital data, in particular of images. This because of the “regularizing effect” due to its parabolic nature.

Another interesting feature of this flow is its connection with certain reaction–diffusion equations, for instance

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon} W'(u),$$

where  $W(u) = (u^2 - 1)^2$  (double–well potential). One can study the singular limits of the solutions of this parabolic equation when  $\varepsilon$  tends to zero. Under suitable hypotheses, it can be shown that the solutions  $u_\varepsilon$  with common initial data converge as  $\varepsilon \rightarrow 0$  to functions which assume only the values  $\pm 1$  in regions separated by boundaries evolving by mean curvature (see [2, 21]).

Further motivation for the study of the mean curvature flow comes from geometric applications, in analogy with the Ricci flow of metrics on abstract Riemannian manifolds. One can use this flow as a tool to obtain classification results for hypersurfaces satisfying certain curvature conditions, to derive isoperimetric inequalities or to produce minimal surfaces. As in Hamilton’s program for the Ricci flow, a fundamental step in order to apply these techniques is the definition of a flow with surgeries or of a generalized (weak) notion of flow, allowing the “passage” through the singularities in a controlled way. There has been much work in this direction by means of techniques based on varifolds, level sets, viscosity solutions (see [1, 3, 4, 9, 17]), till the results of Huisken and Sinestrari [16] that provide a surgery procedure well suited for topological conclusions (see also the recent works by Huisken–Brendle [5] and Haslhofer–Kleiner [12]).

There are striking analogies between the Ricci flow and the mean curvature flow. Indeed, many results hold in a similar way for both flows and several ideas have been successfully exported from one context to the other. However, at the moment no formal way of transforming one of them into the other is known.

In these lectures, I will present exclusively the “classical” parametric setting, without discussing the contributions, sometimes quite relevant, coming from other approaches, in particular, the geometric measure theory setting (see [4, 17]) and the level sets formulation (see [6, 9, 20, 22, 24, 25]).

All the manifolds, quantities and other objects we will consider are smooth, unless otherwise stated. The main tool for the analysis will be a priori estimates (pointwise and integral), very often based on a smart use of the maximum principle in the same spirit of the work of Hamilton for the Ricci flow.

### Further Literature

We definitely suggest to the reader the wonderful survey of White [23] for a general overview of the field.

An excellent introduction to the mean curvature flow is provided in the monograph by Ecker [8], where many basic results and examples are collected. The second part of the book gives a fairly elementary approach to the difficult field of the regularity theory for weak solutions and, in the author’s opinion, it is the natural “next step” for the interested reader. Other nice general references are [7, 14, 18, 26].

Two papers which contain a survey of results on the formation of singularities for mean curvature flow (and also discuss several other geometric flows) are the ones by Huisken [13]

and by Huisken and Polden [15]. It is also surely recommendable to read Sections 2 and 3 of Hamilton's fundamental paper [11]. Such paper deals with the Ricci flow, but many of the ideas there exposed apply to the mean curvature flow as well.

Two works of central importance on weak solutions are the pioneering monograph by Brakke [4] and the memoir by Ilmanen [17]; they are of more difficult reading for a beginner.

Another introductory exposition of the mean curvature flow, including topics not treated in the present notes such as the connection with reaction–diffusion equations, is the one by Ambrosio [2]. The monograph by Giga [10] is also very pleasant to read and it gives a detailed account of the level sets approach to geometric evolutions.

## References

1. F. J. Almgren, J. E. Taylor, and L. Wang, *Curvature driven flows: a variational approach*, SIAM J. Control Opt. **31** (1993), 387–438.
2. L. Ambrosio, *Geometric evolution problems, distance function and viscosity solutions*, Calculus of variations and partial differential equations (Pisa, 1996), Springer–Verlag, Berlin, 2000, pp. 5–93.
3. L. Ambrosio and H. M. Soner, *A level set approach to the evolution of surfaces of any codimension*, J. Diff. Geom. **43** (1996), 693–737.
4. K. A. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, NJ, 1978.
5. S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in  $\mathbb{R}^3$* , ArXiv Preprint Server – <http://arxiv.org>, 2013.
6. Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), 749–786.
7. K. Ecker, *Lectures on geometric evolution equations*, Instructional Workshop on Analysis and Geometry, Part II (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, pp. 79–107.
8. ———, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.
9. L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Diff. Geom. **33** (1991), 635–681.
10. Y. Giga, *Surface evolution equations. A level set approach*, Monographs in Mathematics, vol. 99, Birkhäuser, Basel, 2006.
11. R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7–136.
12. R. Haslhofer and B. Kleiner, *Mean curvature flow with surgery*, ArXiv Preprint Server – <http://arxiv.org>, 2014.
13. G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Proc. Sympos. Pure Math **54** (1993), 175–191.
14. ———, *Lectures on geometric evolution equations*, Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991), Int. Press, Cambridge, MA, 1997, pp. 117–143.
15. G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Springer–Verlag, Berlin, 1999, pp. 45–84.
16. G. Huisken and C. Sinestrari, *Mean curvature flow with surgeries of two–convex hypersurfaces*, Invent. Math. **175** (2009), no. 1, 137–221.
17. T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc., vol. 108(520), Amer. Math. Soc., 1994.
18. ———, *Lectures on mean curvature flow and related equations*, <http://www.math.ethz.ch/~ilmanen/papers/notes.ps>, 1998.
19. W. M. Mullins, *Two–dimensional motion of idealized grain boundaries*, J. Appl. Phys. **27** (1956), 900–904.
20. S. Osher and J. Sethian, *Fronts propagating with curvature–dependent speed: algorithms based on Hamilton–Jacobi formulations*, J. Comput. Phys. **79** (1988), no. 1, 12–49.
21. P. E. Souganidis, *Front propagation: theory and applications*, Viscosity solutions and applications (Montecatini Terme, 1995), Lect. Notes in Math., vol. 1660, Springer–Verlag, Berlin, 1997, pp. 186–242.
22. B. White, *The size of the singular set in mean curvature flow of mean–convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695 (electronic).

23. ———, *Evolution of curves and surfaces by mean curvature*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 2002, pp. 525–538.
24. ———, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), no. 1, 123–138 (electronic).
25. ———, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.
26. X.-P. Zhu, *Lectures on mean curvature flows*, AMS/IP Studies in Advanced Mathematics, vol. 32, Amer. Math. Soc., Providence, RI, 2002.

(Carlo Mantegazza) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI NAPOLI FEDERICO II, VIA CINTIA,  
MONTE S. ANGELO, NAPOLI, ITALY, I-80126

*E-mail address*, C. Mantegazza: `c.mantegazza@sns.it`