Report on the activity carried out in the first 6 months within the Research Grant "Matrix Analysis, Algorithms and Applications" at Dipartimento di Matematica, Università di Pisa

In the first six months of the activity, we have investigated the extension of the concepts of Geometric Mean to the class of positive definite Quasi-Toeplitz (QT) matrices. This class of matrices was the main goal of the program of the project.

More precisely, we have studied means of geometric type of quasi-Toeplitz matrices, that are semi-infinite matrices $A = (a_{i,j})_{i,j=1,2,\ldots}$ of the form $A = T(a) + E$, where E represents a compact operator, and $T(a)$ is a semi-infinite Toeplitz matrix associated with the function a, with Fourier series $\sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell t}$, in the sense that $(T(a))_{i,j} = a_{j-i}$. If a is real valued and essentially bounded, then these matrices represent bounded self-adjoint operators on ℓ^2. We consider the case where a is a continuous function, where quasi-Toeplitz matrices coincide with a classical Toeplitz algebra, and the case where a is in the Wiener algebra, that is, has absolutely convergent Fourier series.

We have proved that if $a_1, \ldots, a_p$ are continuous and positive functions, or are in the Wiener algebra with some further conditions, then means of geometric type, such as the ALM, the NBMP and the Karcher mean of quasi-Toeplitz positive definite matrices associated with $a_1, \ldots, a_p$, are quasi-Toeplitz matrices associated with the geometric mean $(a_1 \cdots a_p)^{1/p}$, which differ only by the compact correction. We have shown by means of numerical tests that these operator means can be practically approximated.

By using functional calculus, we have extended the main matrix functions $f(x)$ of interest, such as square root, p-root, logarithm and esponential, to the case of QT matrices, provided that the spectrum of the matrix is in the domain of definition of $f(x)$.

Then the attention has been addressed to the Karcher mean and to the power mean of positive definite QT matrices. After showing that the power mean of quasi-Toeplitz matrices is a quasi-Toeplitz matrix, we have obtained a first algorithm based on the fact that the Karcher mean is the limit of a family of power means. A second algorithm, that is shown to be more effective, is based on a generalization to the infinite- dimensional case of a reliable algorithm for computing the Karcher mean in the finite-dimensional case. Numerical tests show that the Karcher mean of infinite-dimensional quasi-Toeplitz matrices can be effectively approximated with a finite number of parameters.

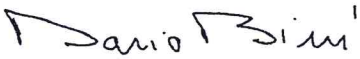
The result of this research collaboration has led to two papers:

D.A. Bini, B. Iannazzo, J. Meng, "Geometric means of quasi-Toeplitz matrices", arXiv:2102.04302, 2021, submitted for publication to an international journal

D.A. Bini, B. Iannazzo, J. Meng, "Algorithms for approximating means of semi-infinite quasi-Toeplitz matrices", submitted to an international conference.

The research will continue on the analysis of iterative methods for the numerical computation of the eigenvalues of QT matrices

Dario A. Bini



Jie Meng

Pisa, March 3, 2021