Divisibility questions in commutative algebraic groups

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Abstract

Let k be a number field and let $\mathcal{A}$ be a commutative algebraic group defined over k. We denote by M_k the set of places $v \in k$, by k_v the completion of k at the valuation v and by G_k the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$.

Problem ((Dvornicich, Zannier, 2001))**.** *Suppose that for all but finitely many $v \in M_k$, there exists $D_v \in \mathcal{A}(k_v)$ such that $P = qD_v$, where P is a fixed k-rational point of $\mathcal{A}$ and q is a fixed positive integer. Is it possible to conclude that there exists $D \in \mathcal{A}(k)$ such that $P = qD$?*

This problem, known as Local-Global Divisibility Problem, arose as a generalization of a particular case of the famous Hasse Principle on quadratic forms. It is linked to another question originary posed by Cassels in 1962, concerning the divisibility of elements of the Tate-Shafarevich group $Ш(k, \mathcal{A})$ in the Weil-Châtelet group $H^1(G_k, \mathcal{A})$. We describe the connection between the two problems and we give an overview of the classical solutions and the main results achieved in the last fifteen years when Cassels' question was taken up again in a more general setting.

Moreover we present the most recent result about the Local-Global Divisibility Problem, showing some sufficient condition to have both an affirmative answer for the local-global divisibility by p in $\mathcal{A}$ and the triviality of $Ш(k, \mathcal{A}[p])$, where $\mathcal{A}[p]$ denotes the p-torsion subgroup of $\mathcal{A}$. If $\mathcal{A}$ is an abelian variety principally polarized, the vanishing of $Ш(k, \mathcal{A}[p])$ implies a local-global principle for divisibility by p for the elements of $H^r(k, \mathcal{A})$, for all $r \geq 0$, giving an answer to a particular case of Cassels' question.