On the Galois Lehmer's problem

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Let α be a non zero algebraic number of degree d which is not a root of unity. We denote by $h(\alpha)$ the usual logarithmic Weil height which is, by a well-know result of Kronecker, > 0. In 1933 Lehmer asked ("Lehmer Problem") if there exists a positive real number c such that $h(\alpha) > cd^{-1}$. This should be the best possible lower bound for the height (without any further assumption on α), since $h(2^{1/d}) = (\log 2)d^{-1}$. The best known result in the direction of Lehmer Problem is Dobrowolski's lower bound, which implies that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that $h(\alpha) \geq c(\varepsilon)d^{-1-\varepsilon}$.

For some infinite extensions $K \subseteq \overline{\mathbb{Q}}$ of the rational field, we know even more than Lehmer, namely an absolute lower bound $h(\alpha) \geq c > 0$ which holds for all $\alpha \in K$. Examples of these extensions are fields with bounded local degrees at some finite place (Bombieri-Zannier 2001) and the maximal abelian extension of $\mathbb{Q}$ (A.-Dvornicich 2000).

Assume now that α is *a generator of a Galois extension* of degree d. We show that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that $h(\alpha) \geq c(\varepsilon)d^{-\varepsilon}$. It is not clear if in this situation we could hope to the stronger lower bond $h(\alpha) \geq c$ (with absolute $c > 0$). Note that such a lower bound would immediately imply the quoted result of A.-Dvornicich.