An introduction to fractional calculus: fundamental ideas and numerics

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Course 30 hrs - 15 Lectures

Fractional calculus, i.e., the definition of integrodifferential operators of non-integer order, has a long history. It runs parallel to that of modern differential calculus as it started with the first correspondence between Leibniz, John Bernoulli, and De l'Hôpital in which it appears as a pure formula-generating curiosity far removed from the geometric interpretation of ordinary derivatives. Then, it evolved towards more modern works by Riemann, Hardy, and Littlewood, finally reaching the contemporary efforts to extend and interpret its objects [5, 6]. Where the possibility of doing rigorous mathematics and proving results within the new theory may be reason enough to invest time in its study, the push of the last few years in this endeavor is due, as often happens, to the birth of different areas of application outside of mathematics. Consider, for example, the modeling of the transport of fluids within highly non-homogeneous media, the modeling of non-local phenomena, or the search for evolutionary models of a non-Markovian nature. These needs have led to the definition of new operators or to the reinterpretation of classical instruments and, at the same time, to the search for numerical methods to simulate these models efficiently and accurately.

The purpose of this course is, on the one hand, to introduce the theoretical and analytical tools necessary to formulate the constitutive laws of these new models (definitions of operators, existence and uniqueness theorems, regularity of the solution, etc.), from other, to show how it is possible to construct numerical methods for their simulation. In particular, we will focus on the definition of fractional integral operators, with the associated fractional derivative operations according to Riemann-Liouville, Caputo, and Riesz. With these instruments we will first discuss the formulation of ordinary fractional equations and their solution [3], then their use in the definition of fractional diffusion problems as a fractional partial differential equation [7]. Regarding numerical methods, we will focus on finite difference and multi-step methods for simulating initial value problems with time-fractional derivatives. Regarding the solution of boundary value problems with space-fractional derivatives, we will focus on the use of finite difference methods and on the connection between the properties of the resulting discretization matrices and the analytic properties of the operators.

In the final part, we will briefly touch on some questions concerning the definition of the fractional Laplacian and some of its applications within modeling for complex networks [1, 2, 4].

Prerequisites. The course makes use of some basic concepts about Lebesgue integration – e.g., $\mathbb{L}^p$ spaces, Fubini's Theorem. Regarding the computational parts, some basic knowledge of quadrature formulas, finite differences for ordinary differential operators, and multi-step methods for initial value problems. For the final parts, it is useful to have some skills with Krylov type projective methods for the solution of linear systems and the concept of preconditioning.

Scheduling. The course will be divided into two parts, the first in May (6 lessons 2h, 3 weeks) and in the months of September/October (9 lessons 2h, 5 weeks).

Modalities. Compatibly with the evolution of the pandemic situation, the rules, and the number of participants, we intend to carry out the course in face-to-face mode. If it is necessary for organizational reasons, it is expected to have it also in streaming mode.

References

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[3] K. Diethelm. *The analysis of fractional differential equations.* Vol. 2004. Lecture Notes in Mathematics. An application-oriented exposition using differential operators of Caputo type. Springer-Verlag, Berlin, 2010, pp. viii+247. isbn: 978-3-642-14573-5. doi: 10.1007/978-3-642-14574-2.

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