RELAZIONE SEMESTRALE

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1. Research projects

1.1. Bounded volume denominators and the generalized Clifford–Severi inequalities. This joint project together with Prof. Rita Pardini aims to elucidate the implications of the generalized Clifford–Severi inequalities developed by Barja, Pardini and Stoppino [2] to the study of volumes of divisors on irregular surfaces.

The notion of volume of a divisor D is a measure of growth of the dimension of sections of multiples of D. Let X be a projective variety of dimension n. Then the volume is defined as

$$\mathrm{vol}_X(D) := \limsup_{m\to\infty} \frac{h^0(\mathcal{O}_X(mD))}{m^n/n!}.$$

The study of Clifford–Severi inequalities, originating from a statement of Severi [14] with incorrect proof which was later proved by Pardini in full generality [29], has recently lead to generalized Clifford–Severi inequalities taking the following form.

Theorem 1. [2] *Let X be a smooth complex projective variety of dimension n, $a : X \to A$ a morphism into an abelian variety such that $\mathrm{Pic}^0(A)$ injects into $\mathrm{Pic}^0(X)$, let $L \in \mathrm{Div}(X)$ and let $h^0_a(L) := \min\{h^0(\mathcal{O}_X(L) \otimes a^*\alpha) \mid \alpha \in \mathrm{Pic}^0(A)\}$. Then*

$$\mathrm{vol}_X(L) \geq n!\, h^0_a(L).$$

In [3] the authors show a connection between properties of the set of volumes of divisors and the bounded negativity conjecture. The boundedness of the denominators of volumes of divisors on a given complex projective surface is shown to be closely related to the so called *bounded negativity conjecture* stating that the self-intersection of all curves on a given surface is bounded below by a constant depending only on the surface.

A weaker conjecture than the bounded volume denominator conjecture is the following conjecture regarding lower bounds of the set of volumes on a surface.

Conjecture 2. Let S be a smooth projective complex surface. Then there exists a constant $C > 0$ depending only on S such that for all big $L \in \mathrm{Div}(S)$ we have $\mathrm{vol}_X(L) \geq C$.

Note that strong statements on the non-vanishing of h^0_a imply the above conjecture for irregular surfaces by the Clifford–Severi inequality.

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Proposition 3. *Let S be a smooth complex projective surface, $a : S \to A$ a morphism into an abelian variety such that $\operatorname{Pic}^0(A)$ injects into $\operatorname{Pic}^0(S)$. Assume that there exists a constant $k > 0$ such that for all but finitely many big divisors L we have that $h^0_a(kL) \neq 0$. Then Conjecture 2 holds true for S.*

This leads to the following general question.

Question 4. *Let X be a smooth complex projective variety of dimension n, $a : X \to A$ a morphism into an abelian variety such that $\operatorname{Pic}^0(A)$ injects into $\operatorname{Pic}^0(X)$, let $L \in \operatorname{Div}(X)$. What are sufficient conditions for $h^0_a(L) \neq 0$?*

The goal of this project is to give (possibly partial) answers to the above question. One possible direction here is to employ the non-vanishing theorems for effective divisors as in [25, 9.4E].

1.2. **Thue–Siegel principle for projective varieties in arbitrary dimension.** The aim of this research project is to produce a Thue–Siegel principle for projective varieties in arbitrary dimension. In the case of projective spaces this can be seen as a first step towards an effective Schmidt subspace theorem. The project combines tools coming from algebraic geometry, stochastics and Diophantine approximations. This is a joint project together with Prof. Alex Küronya and Prof. Catriona Maclean.

Diophantine approximation is an important branch of number theory that is mainly concerned with the study of the quality of approximations of real numbers by rationals. It has strong connections to the theory of Diophantine equations and therefore it is central to many of the most interesting questions in number theory shaping the area today, among them famous open problems like the *abc*-conjecture, which may be seen as a generalization of Wiles's theorem and higher dimensional analogues such as Vojta's conjecture. Unfortunately the most important theorems in Diophantine approximation, for example Roth's theorem [30], which states that for a given irrational algebraic number α and $\epsilon > 0$ there are only finitely many coprime integers $p \in \mathbb{Z}, q \in \mathbb{N}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| \leq q^{-(2+\epsilon)}, \tag{1}$$

still remain ineffective, in that there is no explicit way to bound the absolute value of the solutions of inequality 1. This ineffectivity extends to many applications of Diophantine approximation to Diophantine equations. There are, however, effective methods in Diophantine approximation. To this day, Baker's seminal theorem on linear forms of logarithms [1] provides one of the most important approaches to effectively solve Diophantine equations. Unfortunately this approach is not applicable to the simultaneous approximation of algebraic numbers by rationals. Here one has to rely on Schmidt's generalization of Roth's theorem [33], which is also ineffective. Another effective approach in the approximation of a single algebraic number has been developed by Bombieri [6, 9, 7, 8, 5, 4]. This approach goes back to Thue [35], who proved his theorem on Diophantine approximation using the realization that the existence of one good approximation with large denominator excludes the existence of any other good approximation. This is also the approach of the subsequent developments by Siegel [34], Dyson [15] and finally Roth [30] (in Roth's approach one even needs a large number of good approximations). Thue already noticed that it is possible to make the proof effective if one has a single good approximation with large denominator. However, in the proofs of Thue, Siegel and Roth this denominator needs to be so large that suitable approximations have not been found. Bombieri realized that the approach of Dyson is better suited for this argument and proceeded to give the first examples where this argument, which he named Thue–Siegel principle, gives an effective approximation theorem.

The proof of the theorems of Thue, Siegel, Dyson and Roth all consist of two steps:

1. First an auxiliary polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ having a certain order of vanishing at $(\alpha, \ldots, \alpha)$ is constructed, which is then shown to vanish to a suitable order at $(p_1/q_1, \ldots, p_n/q_n)$ where p_i/q_i are solutions to inequality 1.
2. Next, one shows that there exists an upper bound for the order of P at the point $(p_1/q_1, \ldots, p_n/q_n)$ obtaining a contradiction. This upper bound may be either of geometric (Dyson's lemma [15] or rather its generalization by Esnault and Viehweg [16]) or of arithmetic nature (Roth's lemma [30] and Faltings's product theorem [19]).

The advantage of the geometric approach is that the upper bound for the order of P does not depend on the the point $(p_1/q_1, \ldots, p_n/q_n)$. Therefore, it is enough to find $n-1$ good approximations to effectively bound the denominator of all other good approximations. This is in contrast to the arithmetic approach where $n-1$ good approximations *with large denominators* have to be found. The problem is that no higher dimensional generalization of Dyson's lemma is known.

Positivity concepts for divisors play a crucial role in algebraic geometry. The original and most important notion of positivity is *ampleness*. Its significance comes partially from the fact that it has numerical, cohomological and geometric interpretations. Ampleness is not invariant under birational maps, hence a birational version of it is of great interest. This yields a weaker form of positivity which is called *bigness*: a divisor D is big iff a multiple of it induces a morphism into projective space that is birational onto its image. There is also a cohomological interpretation of bigness: D is big iff D has positive volume. Positivity can also be studied locally: in [13] Demailly introduces an invariant that measures the local positivity of a divisor at a point, the Seshadri constant, in order to study the Fujita conjecture. The behavior of Seshadri constants, in particular finding lower bounds, and their connections to classical problems as Nagata's conjecture have developed into a subject of intensive study.

Newton–Okounkov bodies provide a valuable tool for studying positivity properties of a big divisor L on a complex projective variety X. Motivated by Okounkov's work in representation theory [28], they were defined in [24] and [26] and have developed to a topic of interest in current research. Let d be the dimension of X and let v be a rank d valuation on the function field of X. Then the Newton–Okounkov body of L with respect to v is the convex body

$$\Delta_v(L) := \text{closed convex hull of } \left(\bigcup_{k \in \mathbb{N}} \frac{1}{k} v\left(H^0\left(\mathcal{O}_X(kL)\right)\right) \right) \subset \mathbb{R}^d.$$

Newton–Okounkov bodies have lots of significant applications to combinatorics, representation theory, mathematical physics and Diophantine approximation. In the case of surfaces it is shown in [26] that they are almost rational polygons and can be computed using variation of Zariski decomposition.

One of the the main contributions of the seminal paper of Faltings and Wüstholz [20] is the observation that the constants showing up in Diophantine approximations on projective varieties can be obtained as the expected value of certain random variables coming from filtrations on the graded ring of sections of a divisor. Later [21, 17, 12, 18] showed that these constants can also be obtained via various geometric invariants. More recently, Mckinnon and Roth [27] showed how Diophantine approximation constants measuring how well a given algebraic point on a projective variety can be approximated by rational points are related to volumes of divisors. This result is extended in [22] to the function field case and in [31], [32] and [23] to the more general case where not only points but closed subschemes are approximated.

In the paper [10] the authors develop a theory of concave transforms and Okounkov bodies of filtered linear series and show how the concave transforms determine the volume of the linear series associated to the filtration [10, Theorem 1.11]. One of the aims of this project is to model vanishing along subvarieties in a probabilistic way: in the above setting every filtration on the graded ring of sections of a divisor yields a random variable on the Newton–Okounkov body (the concave transform associated to the filtration). Here already the case where the filtration is given by order of vanishing at a given point on a surface appears to be highly nontrivial. In particular, the expected value of the resulting random variable and its variance seem to contain substantial geometric information. The Chebyshev inequality, which gives a bound on the distribution function of a random variable involving its expected value and covariance matrix, can then be used to imply interesting bounds on the volume function of the linear series involed.

There is another aspect of the stochastic point of view that is needed in Diophantine approximation arguments and is also present in [20] (although it is used for different random variables). The construction of polynomials in many variables or more generally sections of divisors vanishing on products of varieties can be reinterpreted using the above formulation. It turns out that the distribution of the sum of the associated random variables gives us a lower bound on the dimension of the space of sections vanishing on the product. It is an aim of this project to elucidate this specific aspect as a better understanding of the difference between the distribution of the sum of the random variables and the dimension of the space of sections vanishing on the product would enable us to use tools from stochastics such as the Berry–Esseen inequality for the central limit theorem, which would then yield a good estimate of the dimensions in question.

As outlined above, most results on Diophantine approximation rely on the construction of an auxiliary polynomial having a certain order of vanishing or more generally having the property that certain differential operators applied to it vanish at given points. It is a very important point that the vector space of suitable auxiliary polynomials or more generally sections of divisors on a variety can be interpreted as the space of sections of a divisor on a blowup of the variety. This provides a concrete link to positivity in algebraic geometry.

One of the main innovations of this project is to use an approach that grew out of Faltings's proof of the Mordell–Lang conjecture [19] using information on positivity of divisors on blowups to study the vector spaces of suitable auxiliary polynomials. Consider the space V of sections of a divisor having large index at the approximated points and the subspace W of V of sections having small index at a suitably good rational approximation. Faltings's Siegel lemma then provides a section in W with suitably bounded coefficients in $\mathbb{Z}$ under the condition that $\dim(W) < \dim(V)$. It turns out that it is in general hard to show this last condition. Here we put forth two different strategies to solve this problem. One is the study of the sums of the random variables outlined above and the relation to the dimensions in question. Another possibility is to use the approach of Esnault and Viehweg [16] in their proof of Dyson's lemma. It is not clear if the whole proof can be generalized to higher dimensions, however, the main ingredient is a statement about the weak positivity of a certain sheaf and this specific argument has been generalized by Wessler [36]. The strategy is then to proceed as in the proof of Esnault and Viehweg and show that a suitable divisor on a blowup is nef which would then imply the needed inequality $\dim(W) < \dim(V)$ by the asymptotic Riemann–Roch theorem.

The novelty of this approach is that it avoids providing a zero estimate: only a suitable bound on the dimension of the space of sections with given index is needed. As a consequence it is enough to have a partial understanding of the volume function on blowups of the variety in question. Further, this approach is geometric and therefore (contrary to the existing approaches in higher dimension) makes it possible to derive a higher dimensional Thue–Siegel principle of the same quality as in Bombieri's result.

The potential impact of the this project is large in that the number of effective results in Diophantine approximation and Diophantine equations is quite small and any new result would present a breakthrough in this important area of research. Just to name an example, the famous theorem of Faltings regarding the rational points of smooth algebraic curves of genus greater or equal to 2 remains ineffective. Even if one only considers integral points of these curves in an affine open subset (Siegel's theorem), no effective way of obtaining all solutions is known. By work of Corvaja and Zannier [11] it is known that Siegel's theorem can be proven using the Schmidt subspace theorem on simultaneous approximations. It is therefore a very important problem to extend our knowledge on effective approximation in higher dimension with the implication that this will also extend our understanding of effectivity in Diophantine equations.

2. Research Activities

2.1. **Resarch Talks.** I gave the talk "Local positivity and effective Diophantine Approximation" at Università di Pisa.

2.2. **Conferences.** I participated at the conferences "Classical Elegance: the Geometry of Algebraic Varieties" in Cortona and "La grandezza dei punti piccoli" at Università di Pisa.

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